Exercise 1. [/ 12 pts]

In that exercise the "formulaire" is not allowed.

a) Determine an antiderivative of $f(x) = \frac{\ln(2x)}{x}$

b) Determine an antiderivative of $f(x) = xe^{-2x}$

c) We consider $A = \int f(x) \cdot e^{-x^2} dx$.
Give a function f such that A can be determined with the *integration by substitution method*.
But don't integrate it.

d) Determine an antiderivative of $f(x) = \frac{4x^2+3}{x^2+2x+1}$

a) Sub: $t = \ln(2x)$ $dt = \frac{2}{2x} = \frac{1}{x}dx$ $F(x) = \int t \cdot dt = \frac{1}{2}t^2 = \frac{1}{2}\ln^2(2x)$

b) $\int xe^{-2x}dx = -\frac{x}{2}e^{-2x} - \int -\frac{1}{2}e^{-2x}dx = -\frac{x}{2}e^{-2x} + \frac{1}{2}\int e^{-2x}dx$

$\boxed{\begin{array}{ll} u = -\frac{1}{2}e^{-2x} & u' = e^{-2x} \\ v = x & v' = 1 \end{array}}$

$= -\frac{x}{2}e^{-2x} + \frac{1}{2} \cdot -\frac{1}{2}e^{-2x}$

$= -\frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x} = e^{-2x}\left(-\frac{x}{2} - \frac{1}{4}\right)$

$= \frac{e^{-2x}}{4} \cdot (-2x-1)$

c) For instance $f(x) = -2x$

d) $\frac{4x^2+3}{x^2+2x+1} = 4 + \frac{-8x-1}{x^2+2x+1}$

$$\begin{array}{r|l} 4x^2+3 & x^2+2x+1 \\ -(4x^2+8x+4) & 4 \\ \hline -8x-1 & \end{array}$$

$\int \frac{-8x-1}{x^2+2x+1}dx = -8 \cdot \ln|x+1| - \frac{7}{x+1} + C$

$x^2+2x+1 = (x+1)^2$ One root: $x_0 = -1$

Case #2 α, β : $-8x-1 = \alpha(x+1) + \beta$

$a = 1$ $\begin{cases} -8 = \alpha \\ -1 = \alpha + \beta \end{cases}$ $\beta = -1-(-8) = 7$

So $F(x) = 4x - 8 \cdot \ln(x+1) - \frac{7}{x+1}$

Exercise 2. [/ 7 pts]

a) Determine by computations the four first non zero terms of Taylor's expansion of $f(x) = \ln(1+x)$ around $x = 0$ (MacLaurin : $a = 0$).

b) Determine the value of $A = \lim_{x \to 0} \frac{x \cdot \ln(1+x)}{1-\cos(x)}$ by using Taylor's series (*formulaire* allowed)

a)

$$\left.\begin{array}{lll} y = \ln(1+x) & y(0) = 0 & C_0 = 0 \\ y' = 1/_{1+x} & y'(0) = 1 & C_1 = 1 \\ y'' = -1 \cdot (1+x)^{-2} & y''(0) = -1 & C_2 = -\frac{1}{2!} = -\frac{1}{2} \\ y''' = 2 \cdot (1+x)^{-3} & y'''(0) = 2 & C_3 = \frac{2}{3!} = \frac{1}{3} \\ y^{(4)} = -6(1+x)^{-4} & y^{(4)}(0) = -6 & C_4 = \frac{-6}{4!} = -\frac{1}{4} \end{array}\right\} \ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

b)

$$\frac{x \cdot \ln(1+x)}{1-\cos(x)} = \frac{x^2 - x^3/_2 + x^4/_3 - x^5/_4 \dots}{1 - \left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right)} = \frac{\cancel{x^2}\left(1 - x/_2 + \dots\right)}{\cancel{x^2} \cdot \left(\frac{1}{2} - x^2/_{4!} + \dots\right)}$$

$$\lim_{x \to 0} \dots = \lim_{x \to 0} \frac{1 - x/_2 + \dots}{\frac{1}{2} - x^2/_{4!} + \dots} = \frac{1}{1/_2} = 2$$

Exercise 3. [/ 5 pts]

Say why $I = \int_0^1 \frac{1}{\sqrt[4]{2-2x}} dx$ is an improper integral and determine, if it exists, its value.

$f(x) = \frac{1}{\sqrt[4]{2-2x}} = (2-2x)^{-1/4}$

Domain: I want $2-2x > 0$
$2x < 2$
$x < 1$

Improper as $1 \notin D$

$F(x) = \frac{-1}{2} \cdot \frac{4}{3} (2-2x)^{3/4} = -\frac{2}{3} \cdot (2-2x)^{3/4}$

$= -\frac{2}{3} \cdot \sqrt[4]{2-2x}^3$

$I = \lim_{b \to 1} \underbrace{-\frac{2}{3} \cdot \sqrt[4]{2-2b}^3}_{F(b)} - \underbrace{-\frac{2}{3} \cdot \sqrt[4]{2-0}^3}_{F(0)} = \text{"}\underbrace{-\frac{2}{3} \cdot \sqrt[4]{0}}_{0}\text{"} + \frac{2}{3} \cdot \sqrt[4]{2}^3 \cong 1{,}12$

Exercise 4. [/ 5 pts]

Approximations of π can be obtained by using the function $f(x) = \arctan(x)$ and complex numbers. Compute $(4+i) \cdot (5+3i)$ and determine from that result an approximation of π rounded to 3 digits by using the polynomial approximation of degree 5 for $\arctan(x)$

Clearly justify your computations.

$\arctan(x) \cong x - \frac{x^3}{3} + \frac{x^5}{5} = p_5(x)$

$\underbrace{(4+i)}_{\sqrt{17}\, cis(\varphi_1)} \cdot \underbrace{(5+3i)}_{\sqrt{34}\, cis(\varphi_2)} = 17 + 17i = \sqrt{578} \cdot cis(\underbrace{45°}_{\varphi})$

$\varphi_1 = \arctan\left(\frac{1}{4}\right) \quad \varphi_2 = \arctan\left(\frac{3}{5}\right) \quad \varphi = 45° = \frac{\pi}{4}$ rad

$\underline{\varphi_1 + \varphi_2 = \varphi}$ So $\frac{\pi}{4} = \arctan\left(\frac{1}{4}\right) + \arctan\left(\frac{3}{5}\right)$

$\frac{\pi}{4} \cong \left(\frac{1}{4} - \frac{(1/4)^3}{3} + \frac{(1/4)^5}{5}\right) + \left(\frac{3}{5} - \frac{(3/5)^3}{3} + \frac{(3/5)^5}{5}\right) \cong 0{,}244987 + 0{,}543552$

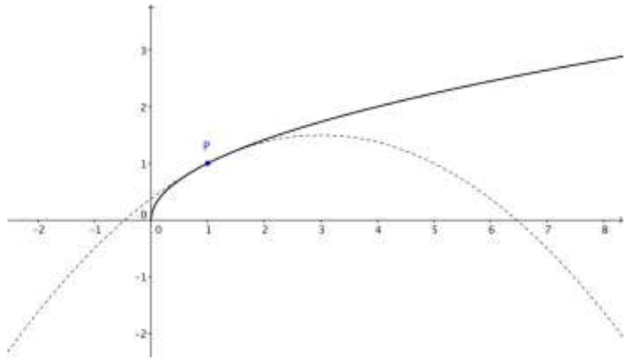
$\cong 0{,}78854$

So $\pi \cong \underline{3{,}154}$

Exercise 5. [/ 6 pts]

a) We consider the function $f(x) = \sqrt{x}$.
Determine the polynomial of degree 2, in the form of your choice, obtained with Taylor's formula **for $a = 1$**.
It's the dashed parabola represented opposite.

b) What's the estimation of $\sqrt{2}$ obtained with that polynomial?



a) $\underline{x = a = 1}$

$f(x) = \sqrt{x} = x^{1/2}$ $\quad f(1) = 1$ $\quad C_0 = 1$

$f'(x) = \frac{1}{2} x^{-1/2}$ $\quad f'(1) = \frac{1}{2}$ $\quad C_1 = \frac{1}{2}$

$f''(x) = \frac{1}{2} \cdot -\frac{1}{2} \cdot x^{-3/2} = -\frac{1}{4} \cdot x^{-3/2}$ $\quad f''(1) = -\frac{1}{4}$ $\quad C_2 = \frac{-\frac{1}{4}}{2!} = -\frac{1}{8}$

$$\sqrt{x} \cong 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 = p_2(x)$$

b) $\sqrt{2} \cong p(2) = 1 + \frac{1}{2} - \frac{1}{8} = 1{,}375$