LDDR. Niveau 2: TE 38 AL-LIN.

EXERCICE 1

a) $R(0, -90°)$ $e_1 \to -e_2$, $e_2 \to e_1$ $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$v = \begin{pmatrix} x \\ y \end{pmatrix}$ $f(v) = M \cdot v = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix}$

b)

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ou $\begin{pmatrix} x+1 \\ y \end{pmatrix}$ ou $\begin{pmatrix} x^2 \\ 1 \end{pmatrix}$ car $f(0) \neq 0$

EXERCICE 2

a) $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow \dim M_{2\times 2} = 4$

$A = \begin{pmatrix} 2 & 0 \\ 1 & -3 \end{pmatrix}$ $B = \begin{pmatrix} 1 & k \\ 2k & 4 \end{pmatrix}$

1) $\operatorname{tr}(A) = 2 - 3 = -1$

2) B singular $\Rightarrow \det B = 0 \Rightarrow 4 - 2k^2 = 0 \Rightarrow k^2 = 2$

$\Rightarrow k = \pm\sqrt{2}$

3) $AB = \begin{pmatrix} 2 & 0 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 7 & -13 \end{pmatrix}$

4) ${}^tA = \begin{pmatrix} 2 & 1 \\ 0 & -3 \end{pmatrix}$

5) the subspace generated by A and has dimension 2 because $A \neq m \cdot B$

$\Rightarrow$ A, B are lin-ind

So we are looking for a matrix C lin dep with the other two

ex $C = 2A$, $C = A + B$ etc.

b) Subspace F $\begin{cases} v_1, v_2 \in F \Rightarrow v_1 + v_2 \in F \\ \lambda v_1 \in F \quad \forall \lambda \in \mathbb{R} \end{cases}$

$M_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ $M_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$

with $a_1 + d_1 = \text{even}$ with $a_2 + d_2 = \text{even}$

- $M_1 + M_2 = \begin{pmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{pmatrix}$

$Tr(M_1 + M_2) = a_1 + a_2 + d_1 + d_2 = \underbrace{(a_1+d_1)+(a_2+d_2)}_{\text{even + even}} =$

$= \text{even}.$

- $\lambda \in R.$ $\lambda M_1 = \begin{pmatrix} \lambda a_1 & \lambda b_1 \\ \lambda c_1 & \lambda d_1 \end{pmatrix}$

$Tr(\lambda M_1) = \lambda(a_1+d_1)$ not even $\forall \lambda \in R$

ex $M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $Tr(M_1) = 2$

$\lambda = 1.5$ $\lambda M_1 = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix}$ $Tr(\lambda M_1) = 3!!$

So **No**

c) M symetric:

$M = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

basis.

$\dim M = 3$

d) $f: M \to tr(M)$

i) ex $M_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

We have to prove that

$M_1 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ $M_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ $M_3 = \begin{pmatrix} 0 & 0 \\ -3 & 0 \end{pmatrix}$ and M_4

are lin ind

Definition: $xM_1 + yM_2 + zM_3 + kM_4 = 0$

$\Rightarrow x = y = z = k = 0$

$x\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} + y\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + z\begin{pmatrix} 0 & 0 \\ -3 & 0 \end{pmatrix} + k\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow \begin{cases} 2x = 0 \\ y = 0 \\ -3z = 0 \\ y + k = 0 \end{cases} \Rightarrow \begin{matrix} x = 0 \\ y = 0 \\ z = 0 \\ k = 0 \end{matrix}$ $\Rightarrow$ lin ind

$\Rightarrow M_1, M_2, M_3, M_4$ basis of $M_{2\times 2}$

ii) We are looking for the images of M_1, M_2, M_3, M_4 under f.

$M_1 \to Tr(M_1) = 2 \quad M_2 \to Tr(M_2) = 1$

$M_3 \to Tr(M_3) = 0 \quad M_4 \to Tr(M_4) = 1$

so $\mathbb{F} = (2 \ 1 \ 0 \ 0)$

EXERCICE 3

$\dim F = 0$ if $v_1 = v_2 = v_3 = 0 \Rightarrow$ point

$\dim F = 1$ if $v_1 // v_2 // v_3$ $\det(v_1, v_2, v_3) = 0 \Rightarrow$ ligne

$\dim F = 2$ if $\det(v_1, v_2, v_3) = 0$ but $v_1 \not\parallel v_2$ $\Rightarrow$ plan

$\dim F = 3$ if $\det(v_1, v_2, v_3) \neq 0 \Rightarrow V_3$

EXERCICE 4

a) A $\vec{v} \neq 0$ so that $\exists\, \lambda \in \mathbb{R}$: $M \cdot v = \lambda v$.

b) $\vec{v} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ $M = \begin{pmatrix} -6 & 4 \\ 2 & k \end{pmatrix}$

$M \cdot v = \lambda v \Rightarrow \begin{pmatrix} -6 & 4 \\ 2 & k \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \lambda \begin{pmatrix} 4 \\ 3 \end{pmatrix} \Rightarrow$

$\Rightarrow \begin{cases} -24 + 12 = 4\lambda \Rightarrow -12 = 4\lambda \Rightarrow \lambda = -3 \\ 8 + 3k = 3\lambda \Rightarrow 8 + 3k = -9 \Rightarrow k = -\frac{17}{3} \end{cases}$

c) $F = \begin{pmatrix} 37 & -16.5 \\ 88 & -40 \end{pmatrix}$

$\det(F - \lambda I) = 0 \Rightarrow \begin{vmatrix} 37-\lambda & -16.5 \\ 88 & -40-\lambda \end{vmatrix} = 0$

$\Rightarrow (37-\lambda)(-40-\lambda) + 1452 = 0 \Rightarrow$

$-1480 + 40\lambda - 37\lambda + \lambda^2 + 1452 = 0 \Rightarrow$

$\lambda^2 + 3\lambda - 28 = 0 \quad \lambda_1 = -7$

d) $G = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 4 & 5 \end{pmatrix}$ triangular matrix $\Rightarrow$ $\lambda_1 = 1 \quad \lambda_2 = 3 \quad \lambda_3 = 5$

$e_3 \rightarrow 5\, e_3$ so $\vec{e}_3$ is 5-eigenvector

EXERICE 5

$F = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

a) $v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad v' = f(\vec{v}) = F \cdot v = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix}$

b) $\mathcal{B} = (\vec{a}, \vec{b}, \vec{c}) \qquad \mathcal{B}' = (u_1, u_2, u_3)$

$\vec{a} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \vec{b} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} c = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad u_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad u_2 = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} \quad u_3 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$

$P = \begin{pmatrix} 0 & 5 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \det P = -5 \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = -5$

$P_{11} = 0 \quad P_{12} = 1 \quad P_{13} = 0$

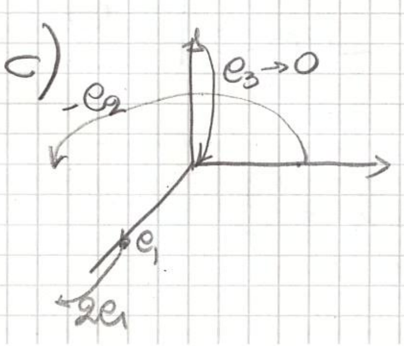
$P_{21} = 0 \quad P_{22} = 0 \quad P_{23} = -5$

$P_{31} = -5 \quad P_{32} = 0 \quad P_{33} = 0$

$P^{-1} = \frac{1}{-5} \begin{pmatrix} 0 & 0 & -5 \\ -1 & 0 & 0 \\ 0 & 5 & 0 \end{pmatrix} =$

$F' = P^{-1} F P = \begin{pmatrix} 0 & 0 & -1 \\ -1/5 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 5 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 & 0 \\ -2/5 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 5 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

c) 

Proj into plan $(a,b) \circ S(\text{axis } a) \circ H(2)$

EXERCICE 6.

$\Pi: 2x - y + 3z = 0 \qquad \vec{n} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \quad \|\vec{n}\| = \sqrt{14}$

$$e_2' \to e_2 - \frac{\vec{e}_2 \cdot \vec{n}}{\|\vec{n}\|^2}\vec{n} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \frac{-1}{14}\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1/7 \\ -1/14 \\ 3/14 \end{pmatrix} = \begin{pmatrix} 1/7 \\ 13/14 \\ 3/14 \end{pmatrix}$$

$\lambda_1 = 1 \;\to$ plan $\vec{P}_1, \vec{P}_2 \perp \vec{n} \Rightarrow \vec{P}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \; P_2 = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$

$\lambda_2 = 0 \;\to$ ligne $\perp$ plan $\vec{P}_3 = \vec{n} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

$$M' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{(P_1, P_2, P_3)}$$