LDDR - Niveau 2 : TE 21 AL - Lin.

EXERCICE 1

1) $M = \begin{pmatrix} 6 & k \\ k & -1 \end{pmatrix}$

$|M - \lambda I| = 0 \Rightarrow \begin{vmatrix} 6-\lambda & k \\ k & -1-\lambda \end{vmatrix} = 0 \Rightarrow$

$\Rightarrow -6 + \lambda - 6\lambda + \lambda^2 - k^2 = 0 \Rightarrow \lambda^2 - 5\lambda - 6 - k^2 = 0$

$\Delta > 0 \Rightarrow 25 - 4(-6 - k^2) > 0 \Rightarrow 25 + 24 + 4k^2 > 0$

$\Rightarrow 49 + k^2 > 0 \quad \forall k \in \mathbb{R}$

2) $v = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ We have $M \cdot v = \lambda v \Rightarrow$

$\Rightarrow \begin{pmatrix} 6 & k \\ k & -1 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \lambda \begin{pmatrix} 4 \\ 3 \end{pmatrix} \Rightarrow \begin{cases} 24 + 3k = 4\lambda & *(-3) \\ 4k - 3 = 3\lambda & * 4 \end{cases}$

$+\begin{cases} -72 - 9k = -\cancel{12\lambda} \\ -12 + 16k = \cancel{12\lambda} \end{cases}$

$-84 + 7k = 0 \Rightarrow 7k = 84 \Rightarrow k = 12$

$24 + 36 = 4\lambda \Rightarrow 4\lambda = 60 \Rightarrow \lambda = 15$

EXERCICE 2

$\vec{a} = \begin{pmatrix} 4 \\ -3 \end{pmatrix} \Rightarrow \vec{n} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \Rightarrow 3x + 4y = 0$

1) $R(0, +180)$ $\quad R_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \quad \varphi = 180$

$R = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

2) $S(\vec{a})$

i) First methode $\quad \vec{v} \to 2 \dfrac{\vec{v} \cdot \vec{d}}{\|d\|^2} \cdot \vec{d} - \vec{v}$

$\vec{d} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$

$e_1 \to 2 \dfrac{1 \cdot 4 + 0(-3)}{\sqrt{16+9}^{\,2}} \cdot \begin{pmatrix} 4 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2 \cdot \dfrac{4}{25} \begin{pmatrix} 4 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$e_1' = \begin{pmatrix} 32/25 \\ -24/25 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow e_1' = \begin{pmatrix} 7/25 \\ -24/25 \end{pmatrix}$

$$C_2 \to 2 \cdot \frac{-3}{25}\begin{pmatrix}4\\-3\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-24/25\\18/25\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix} \Rightarrow$$

$$C_2' = \begin{pmatrix}-24/25\\-7/25\end{pmatrix} \qquad S = \begin{pmatrix}7/25 & -24/25\\-24/25 & -7/25\end{pmatrix}$$

ii) Second

$S = \begin{pmatrix}a & b\\c & d\end{pmatrix}$ $\quad \vec{a}$ invariant under S $\vec{a} \to \vec{a}$

$\vec{n} \perp a$, $\vec{n} = \begin{pmatrix}3\\4\end{pmatrix} \to \begin{pmatrix}-3\\-4\end{pmatrix}$

So, $\quad S \cdot \vec{a} = \vec{a}$ (1)

$\quad S \cdot \vec{n} = -\vec{n}$ (2)

(1) $S\vec{a} = \vec{a} \Rightarrow \begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}4\\-3\end{pmatrix} = \begin{pmatrix}4\\-3\end{pmatrix} \Rightarrow$

$$\begin{cases}4a - 3b = 4\\4c - 3d = -3\end{cases}$$

(2) $S \cdot \vec{n} = -\vec{n} \Rightarrow \begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}+3\\+4\end{pmatrix} = \begin{pmatrix}-3\\-4\end{pmatrix} \Rightarrow$

$$\Rightarrow \begin{cases}3a + 4b = -3\\3c + 4d = -4\end{cases}$$

$$\begin{cases}4a - 3b = 4 & \cdot 4\\3a + 4b = -3 & \cdot 3\end{cases} \qquad \begin{cases}16a - 12b = 16\\9a + 12b = -9\end{cases}$$

$$+ \quad 25a = 7 \Rightarrow a = 7/25$$

$$4 \cdot \frac{7}{25} - 3b = 4 \Rightarrow \frac{28}{25} - 4 = 3b \Rightarrow$$

$$\Rightarrow \frac{-72}{25} = 3b \Rightarrow b = -\frac{24}{25}$$

$$\begin{cases}4c - 3d = -3 & \cdot 4\\3c + 4d = -4 & \cdot 3\end{cases} \qquad \begin{cases}16c - 12d = -12\\9c + 12d = -12\end{cases}$$

$$25c = -24 \Rightarrow c = -\frac{24}{25}$$

$$4\left(-\frac{24}{25}\right) - 3d = -3 \Rightarrow -\frac{96}{25} + 3 = 3d \Rightarrow$$

$$-\frac{21}{25} = 3d \Rightarrow d = -\frac{7}{25}$$

So $S = \begin{pmatrix} 7/25 & -\frac{24}{25} \\ -\frac{24}{25} & -7/25 \end{pmatrix}$

3) i) $\vec{v} \rightarrow \frac{\vec{v}\cdot\vec{d}}{\|d\|^2}\cdot\vec{d}$ $\quad \vec{d} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$

$v = e_1$

$e_1 \rightarrow \frac{4}{25}\begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 16/25 \\ -12/25 \end{pmatrix}$ $\quad R = \begin{pmatrix} 16/25 & -12/25 \\ -12/25 & 9/25 \end{pmatrix}$

$e_2 \rightarrow \frac{-3}{25}\begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} -12/25 \\ 9/25 \end{pmatrix}$

ii) <u>Or</u>

$R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad \vec{a}$ = invariant

$\vec{n} \perp \vec{a} \quad \vec{n} \rightarrow 0$

$R\cdot\vec{a} = \vec{a} \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix} \Rightarrow$

$\begin{cases} 4a - 3b = 4 \\ 4c - 3d = -3 \end{cases}$

$R\vec{n} = 0 \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$

$\Rightarrow \begin{cases} 3a + 4b = 0 \\ 3c + 4d = 0 \end{cases}$

$\begin{cases} 4a - 3b = 4 & *4 \\ 3a + 4b = 0 & *3 \end{cases}$ $\quad \begin{cases} 16a - 12b = 16 \\ 9a + 12b = 0 \end{cases}$

$25a = 16 \Rightarrow a = 16/25$

$4\cdot\frac{16}{25} - 3b = 4 \Rightarrow \frac{64}{25} - 4 = 3b \Rightarrow$

$\Rightarrow -\frac{36}{25} = 3b \Rightarrow b = -\frac{12}{25}$

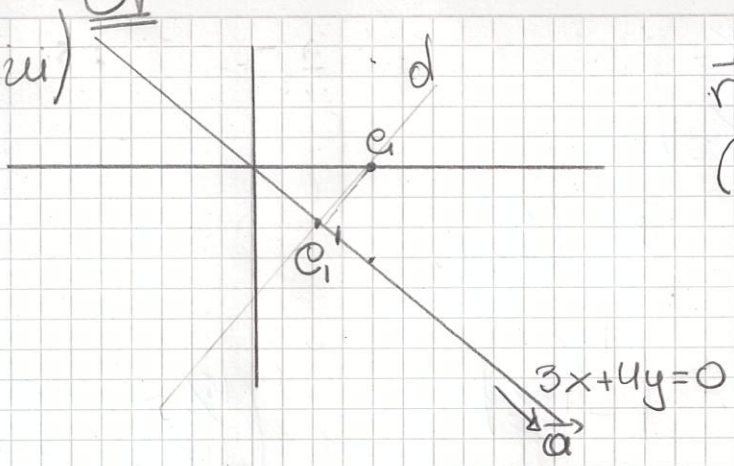
$\begin{cases} 4c - 3d = -3 & *4 \\ 3c + 4d = 0 & *3 \end{cases}$ $\quad \begin{cases} 16c - 12d = -12 \\ 9c + 12d = 0 \end{cases}$

$25c = -12 \Rightarrow c = -\frac{12}{25}$

$4\left(-\frac{12}{25}\right) - 3d = -3 \Rightarrow d = 9/25.$

Or

iii)



$\vec{n}_d = \vec{a} = \begin{pmatrix} 4 \\ -3 \end{pmatrix} \Rightarrow d: 4x-3y+c=0$

$(1,0): 4+c=0 \Rightarrow c=-4$

$d: 4x-3y-4=0$

$e_1' = d \cap a \quad \begin{cases} 3x+4y=0 & *3 \\ 4x-3y=4 & *4 \end{cases} \quad \begin{cases} 9x+12y=0 \\ 16x-12y=16 \end{cases}$

$25x = 16 \Rightarrow x = 16/25$

$3\,\frac{16}{25} + 4y = 0 \Rightarrow 4y = -\frac{48}{24} \Rightarrow y = -\frac{12}{25}$

$e_1' = \begin{pmatrix} 16/25 \\ -12/25 \end{pmatrix}$

the same for e_2'.

4) S symetrie $\Rightarrow \lambda_1 = 1 \quad p_1 = \vec{a} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$

$\lambda_2 = -1 \quad p_2 \perp p_1 \Rightarrow p_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$S' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{(p_1, p_2)}$

5) P projection $\Rightarrow \lambda_1 = 1 \quad \vec{p}_1 = \vec{a} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$

$\lambda_2 = 0 \quad p_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$P' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{(p_1, p_2)}$

b) $P^2 = P \cdot P = P$

$S^3 = S^2 \cdot S = I \cdot S = S$

$SP = P$

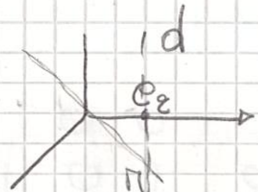
$RP \qquad e_1 \rightarrow -e_1'(pr) \quad RP = -P$

$e_2 \rightarrow -e_2'(pr)$

$$S+I = \frac{1}{25}\begin{pmatrix} 7 & -24 \\ -24 & -7 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 32/25 & -24/25 \\ \frac{-24}{25} & 18/25 \end{pmatrix}$$

EXERCICE 3

$\pi: x+2y-3z=0$



3) e_2' ?

$$\vec{d} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad d = \begin{cases} x = 0 + 0\lambda \\ y = 1 + 0\lambda \\ z = 0 + \lambda \end{cases}$$

$e_2' = d \cap \pi \qquad 0 + 2 - 3\lambda = 0 \Rightarrow \lambda = 2/3$

$$e_2' = \begin{pmatrix} 0 \\ 1 \\ 2/3 \end{pmatrix}$$

4) $\lambda_1 = 1 \rightarrow$ plan $\pi \quad P_1, P_2 \in \pi \Rightarrow P_1, P_2 \perp \vec{n} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$

$$\Rightarrow P_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \quad P_2 = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$$

$$\lambda_2 = 0 \rightarrow e_3 \Rightarrow P_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

EXERCICE 4

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 3 & -2 \end{pmatrix} \qquad \vec{a} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

1) $f(a) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 3 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

2) $f(v) = b \Rightarrow A \cdot v = b \Rightarrow v = A^{-1} \cdot b$

A^{-1}? $\quad \det A = 1 \cdot 1(-2) = -2$

$A_{11} = -2 \qquad A_{12} = 0 \qquad A_{13} = -3$

$A_{21} = 0 \qquad A_{22} = -2 \qquad A_{23} = 3$

$A_{31} = 0 \qquad A_{32} = 0 \qquad A_{33} = 1$

$$A^{-1} = \frac{1}{-2} \,{}^t\begin{pmatrix} -2 & 0 & -3 \\ 0 & -2 & -3 \\ 0 & 0 & 1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ -3 & -3 & 1 \end{pmatrix} \Rightarrow$$

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3/2 & 3/2 & -1/2 \end{pmatrix}$$

$$V = A^{-1} \cdot b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3/2 & 3/2 & -1/2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

3) Invariant vector V $\quad A \cdot V = V \quad \Rightarrow \lambda = 1$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{cases} x = x \\ y = y \\ 3x + 3y - 2z = z \end{cases}$$

$$\Rightarrow x + y - z = 0 \Rightarrow z = x + y$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ x+y \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ x \end{pmatrix} + \begin{pmatrix} 0 \\ y \\ y \end{pmatrix}$$

$$\Rightarrow P_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} ; P_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

EXERCICE 5

$$G = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

1) $$G \cdot G = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -3 & -3 \\ -3 & 3 & 3 \\ -3 & 3 & 3 \end{pmatrix}$$

$$\Rightarrow G^2 = 3G$$

2) $$|G-\lambda I| = \begin{vmatrix} 1-\lambda & -1 & -1 \\ -1 & 1-\lambda & 1 \\ -1 & 1 & 1-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 1-\lambda & -1 & -1 \\ -\lambda & -\lambda & 0 \\ -1 & 1 & 1-\lambda \end{vmatrix} = -1 \begin{vmatrix} -\lambda & -\lambda \\ -1 & 1 \end{vmatrix} + (1-\lambda) \begin{vmatrix} 1-\lambda & -1 \\ -\lambda & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow -(-\lambda-\lambda) + (1-\lambda)(-\lambda+\lambda^2-\lambda) = 0 \Rightarrow$$

$$\Rightarrow 2\lambda + (1-\lambda)(\lambda^2-2\lambda) = 0 \Rightarrow$$

$$\Rightarrow \cancel{2\lambda} + \lambda^2 - \cancel{2\lambda} - \lambda^3 + 2\lambda^2 = 0 \Rightarrow$$

$$\Rightarrow -\lambda^3 + 3\lambda^2 = 0 \Rightarrow -\lambda^2(\lambda-3) = 0$$

3) $\lambda = 0 \quad \lambda = 3$

$$\lambda = 0 \quad \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$$

$$\Rightarrow x - y - z = 0 \Rightarrow x = y + z$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y+z \\ y \\ z \end{pmatrix} = y \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad P_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad P_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda = 3 \quad \begin{pmatrix} -2 & -1 & -1 \\ -1 & -2 & 1 \\ -1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$$

$$\begin{cases} -2x - y - z = 0 \\ -x - 2y + z = 0 \\ -x + y - 2z = 0 \end{cases} \quad + \;\; -3x - 3y = 0 \Rightarrow y = -x$$

$$-x - x - 2z = 0 \Rightarrow z = -x$$

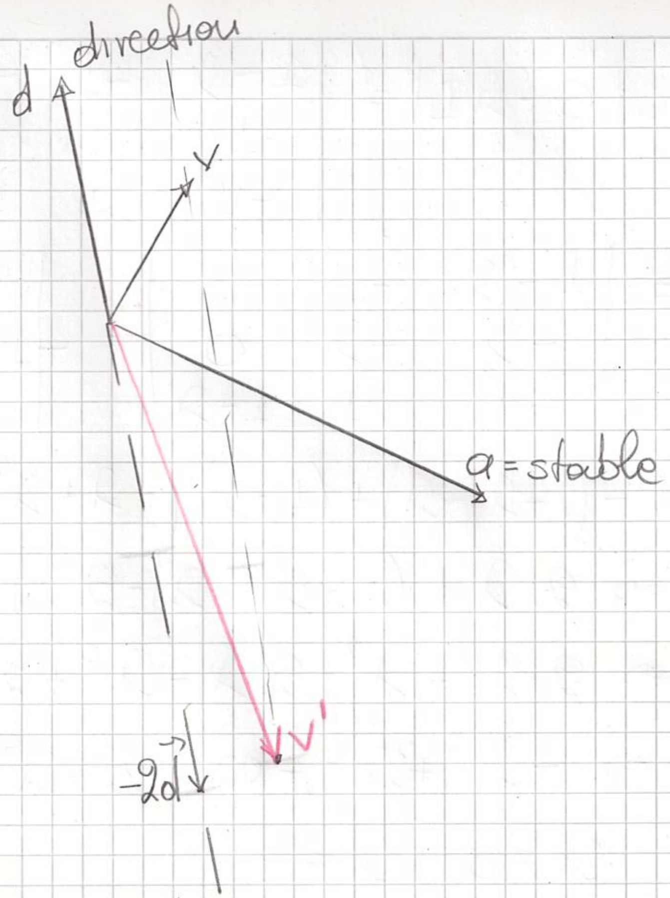
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -x \\ -x \end{pmatrix} \qquad P_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

Projection on the ligne $// p_3$, direction of plane generated by P_1, P_2 (orthogonal)

EXERCICE 6

2) $\vec{d} \to -2d + 0 \cdot \vec{a}$
$\vec{a} \to 0 \cdot d + 1 \cdot \vec{a}$

$$M = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$$



EXERCICE 4

a) $f : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 2y - x \\ 2x \end{pmatrix}$

$V_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad V_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \quad V_1 + V_2 = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$

$$f\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} = \begin{pmatrix} 2(y_1 + y_2) - (x_1 + x_2) \\ 2(x_1 + x_2) \end{pmatrix} = \begin{pmatrix} 2y_1 - x_1 + 2y_2 - x_2 \\ 2x_1 + 2x_2 \end{pmatrix}$$

✓

$$f(V_1) + f(V_2) = \begin{pmatrix} 2y_1 - x_1 \\ 2x_1 \end{pmatrix} + \begin{pmatrix} 2y_2 - x_2 \\ 2x_2 \end{pmatrix} = \begin{pmatrix} 2y_1 - x_1 + 2y_2 - x_2 \\ 2x_1 + 2x_2 \end{pmatrix}$$

$$f(\lambda V_1) = f\begin{pmatrix} \lambda x_1 \\ \lambda y_1 \end{pmatrix} = \begin{pmatrix} 2\lambda y_1 - \lambda x_1 \\ 2x_1 \end{pmatrix}$$

✓

$$\lambda f(V_1) = \lambda \begin{pmatrix} 2y_1 - x_1 \\ 2x_1 \end{pmatrix} = \begin{pmatrix} 2\lambda y_1 - \lambda x_1 \\ 2\lambda x_1 \end{pmatrix}$$

b) $e_1 \to e_2$
$e_2 = -e_1$

$M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$M \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

c) $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x+1 \\ y \end{pmatrix} \quad f(0) \neq 0$