# LDDR Niveau 2: Algebre Lineaire Exercices

## LINEAR ALGEBRA EXERCISES

## 2. Basis. Components. Dimension

1. Let $(\overrightarrow{e_1}\,;\,\overrightarrow{e_2})$ be the standard basis of $V_2$.
   Consider the vector $\vec{v} = 8\overrightarrow{e_1} + 2\overrightarrow{e_2}$ .
   a) Find the components of $\vec{v}$ with respect to the basis $(\overrightarrow{e_1}\,;\,\overrightarrow{e_2})$.
      Similar question with the basis $(\overrightarrow{e_2}\,;\,\overrightarrow{e_1})$.
      Similar question with the basis $(2\overrightarrow{e_1}\,;\,-4\overrightarrow{e_2})$.
   b) Show that the vectors $\overrightarrow{v_1} = \overrightarrow{e_1} - \overrightarrow{e_2}$ and $\overrightarrow{v_2} = \overrightarrow{e_1} + \overrightarrow{e_2}$ form a basis of $V_2$.
   c) Calculate the components of $\vec{v}$ with respect to the basis $(\overrightarrow{v_1}\,;\,\overrightarrow{v_2})$.

2. Let $(\overrightarrow{e_1}\,;\,\overrightarrow{e_2}\,;\,\overrightarrow{e_3})$ be the standard basis of $V_3$.
   Consider the vectors $\overrightarrow{v_1} = 5\overrightarrow{e_1} - 2\overrightarrow{e_2} + \overrightarrow{e_3}$ and $\overrightarrow{v_2} = 3\overrightarrow{e_1} + \overrightarrow{e_2} - 5\overrightarrow{e_3}$ .
   Find the components of $\vec{v} = 18\overrightarrow{e_1} + 20\overrightarrow{e_2} + 24\overrightarrow{e_3}$ with respect to the basis $(\overrightarrow{v_1}\,;\,\overrightarrow{v_2}\,;\,\overrightarrow{v_1} \wedge \overrightarrow{v_2})$.

3. Let $(\overrightarrow{e_1}\,;\,\overrightarrow{e_2}\,;\,\overrightarrow{e_3})$ be the standard basis of $V_3$.
   Consider the vectors $\overrightarrow{v_1} = m\overrightarrow{e_1} + 2\overrightarrow{e_2} + \overrightarrow{e_3}$ , $\overrightarrow{v_2} = 2\overrightarrow{e_1} + m\overrightarrow{e_2} + 2\overrightarrow{e_3}$ and $\overrightarrow{v_3} = \overrightarrow{e_1} + 2\overrightarrow{e_2} + \overrightarrow{e_3}$.
   For what values of $m$ is $(\overrightarrow{v_1}\,;\,\overrightarrow{v_2}\,;\,\overrightarrow{v_3})$ a basis of $V_3$?
   Explain your reasoning.

4. Let $V$ be the vector space of polynomials of degree $\leq 2$.
   a) Find the components of $p = 3x^2 - 4x + 5$ with respect to the basis $(1, x, x^2)$.
   b) Show that the set of polynomials $p_1 = x^2 + 2x$ , $p_2 = x + 1$ and $p_3 = x^2 + 3$ is a basis of $V$.
   c) Find the components of $p = 3x^2 - 4x + 5$ with respect to the basis $(p_1, p_2, p_3)$.

## 3. Subspaces

5. In $V_3$, with the standard basis, consider the subset $W$ of all the vectors of the form $\begin{pmatrix} a \\ b \\ a-b \end{pmatrix}$.

   a) Show that $W$ is a subspace.

   b) Find a basis of $W$.

   c) The subspace $W$ is a vector plane. Find an equation of $W$.

6. In $V_3$ consider the vector plane $W$ with equation is $3x - 2y + 5z = 0$. Find a basis of $W$.

7. In $V_3$, with the standard basis, consider the subspace $W$ of all the linear combinations of $\overrightarrow{v_1} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and $\overrightarrow{v_2} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$. Let $\vec{v} = \begin{pmatrix} 6 \\ m \\ -9 \end{pmatrix}$.

   a) Why is $(\overrightarrow{v_1}\,;\,\overrightarrow{v_2})$ a basis of $W$?

   b) Find $m$ such that $\vec{v} \in W$.

   For the next questions assume that $m = -7$.

   c) Find the components of $\vec{v}$ with respect to the basis $(\overrightarrow{v_1}\,;\,\overrightarrow{v_2})$ of $W$.

   d) Why is $(\overrightarrow{2v_1}\,;\,\overrightarrow{v_2} - \overrightarrow{v_1})$ a basis of $W$?

   e) Find the components of $\vec{v}$ with respect to the basis $(\overrightarrow{2v_1}\,;\,\overrightarrow{v_2} - \overrightarrow{v_1})$ of $W$.

8. In $V_3$, with the standard basis, consider the subspace $W$ of all the linear combinations of the vectors $\overrightarrow{v_1} = \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix}$, $\overrightarrow{v_2} = \begin{pmatrix} 1 \\ -3 \\ -4 \end{pmatrix}$ and $\overrightarrow{v_3} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

   a) Is $(\overrightarrow{v_1}\,;\,\overrightarrow{v_2}\,;\,\overrightarrow{v_3})$ a basis of $W$? Justify the answer.

   b) If $W$ is a vector plane, find a basis of $W$ and an equation of $W$.

9. Let $V$ be the vector space of polynomials of degree $\leq 2$.

Let $m \in \mathbb{R}$ be a fixed number.

a) Show that the set $W$ of all the polynomials which take the value 0 for $x = m$ is a subspace of $V$.

b) Find a basis of $W$.

*Hint*: a polynomial $p$ which is 0 for $x = m$ can be factorized by $x - m$.

10. Consider the vector space $V$ of all linear combinations of the functions $s: x \longmapsto y = \sin(x)$ and $c: x \longmapsto y = \cos(x)$.

a) Show that the set $(s; c)$ is a basis of $V$.

b) With respect to this basis find the components of

$$f: x \longmapsto y = -\cos(x)\,,\ g: x \longmapsto y = \sin^2(x) \text{ and } h: x \longmapsto y = \sin\left(x - \frac{\pi}{4}\right)$$

## 4. Linear transformations

11. In each case determine whether the transformation is linear. The transformations are given with respect to the standard basis.

a) $T: V_2 \to V_2\,, \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2x \\ x - 3y \end{pmatrix}$ b) $T: \mathbb{R} \to V_2\,, x \mapsto \begin{pmatrix} x \\ x^2 \end{pmatrix}$

12. Let $V$ be the vector space of polynomials of degree $\leq 1$ and $m$ be a fixed number. Is the following transformation linear?

$$f: V \to \mathbb{R}\,, p = ax + b \mapsto p(m) = a \cdot m + b$$

## 5. Matrix of a linear transformation

## and

## 6. Components of the image of a vector under a linear transformation

13. Find the matrix, with respect to the standard basis $(\overrightarrow{e_1}\ ;\ \overrightarrow{e_2})$, of each of the following linear transformation of $V_2$.
    a) reflection in the line parallel to $\overrightarrow{e_2}$ (equation: $x = 0$)
    b) reflection in the line $x - y$ (parallel to $\overrightarrow{e_1} + \overrightarrow{e_2}$)
    c) projection onto the line $y = 0$ (parallel to $\overrightarrow{e_1}$)
    d) projection onto the line $3x - 2y = 0$ (normal to $3\overrightarrow{e_1} - 2\overrightarrow{e_2}$)
    e) reflection in the line $3x - 2y = 0$ (parallel to $2\overrightarrow{e_1} + 3\overrightarrow{e_2}$)
    f) rotation through +30°
    g) dilatation of factor $k$
    h) projection in the direction of $\vec{d} = \binom{3}{1}$ onto the line normal to $\vec{n} = \binom{1}{2}$

14. Find the matrix, with respect to the standard basis $(\overrightarrow{e_1}\ ;\overrightarrow{e_2}\ ;\ \overrightarrow{e_3})$, of each of the following linear transformation of $V_3$.
    a) reflection in the plane generated by $\overrightarrow{e_1}$ and $\overrightarrow{e_3}$
    b) reflection in the line parallel to $\overrightarrow{e_2}$
    c) projection onto the line parallel to $\vec{d} = \overrightarrow{e_1} + \overrightarrow{e_2} + \overrightarrow{e_3}$
    d) projection onto the plane $x = 0$ (generated by $\overrightarrow{e_1}$ and $\overrightarrow{e_2}$)
    e) projection onto the plane $x + y - z = 0$ (normal to $\overrightarrow{e_1} + \overrightarrow{e_2} - \overrightarrow{e_3}$)
    f) reflection in the plane $x + y - z = 0$
    g) rotation through an angle 90° about the line parallel to $\overrightarrow{e_3}$ and such that the image of $\overrightarrow{e_1}$ is $\overrightarrow{e_2}$

15. Use the matrices of the preceding exercise to calculate the images of the vector $\vec{v} = 2\overrightarrow{e_1} - 3\overrightarrow{e_2}$ under the transformations d) to h) of exercise 13.

16. Use the matrices of the preceding exercise to calculate the images of the vector $\vec{v} = 2\vec{e_1} - 3\vec{e_2} + 5\vec{e_3}$ under the given transformations c), e), f) and g) of exercise 14.

17. In each case describe geometrically the linear transformation on $V_n$ given by its matrix with respect to the standard basis.

a) $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ b) $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ c) $C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

d) $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ e) $E = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ f) $F = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

g) $G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ h) $H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$ i) $J = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}$

18. Consider the linear transformation $T: V_3 \to V_2, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x+2y \\ z-2y \end{pmatrix}$ given with respect to the standard bases $(\vec{e_1}\,; \vec{e_2}\,; \vec{e_3})$ and $(\vec{f_1}\,; \vec{f_2})$.

a) Find its matrix with respect the standard bases.

b) Use this matrix to find the image of the vector $\vec{v} = 3\vec{e_1} - 4\vec{e_2} - \vec{e_3}$

19. In $V_2$ consider the linear transformation $T: \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 3x+2y \\ -x+4y \end{pmatrix}$ given with respect to the standard basis $(\vec{e_1}\,; \vec{e_2})$.

a) Find the matrix $M$ of $T$ with respect to the standard basis.

b) Let $\vec{v} = 2\vec{e_1} - \vec{e_2}$. Use $M$ to find the components of $T(\vec{v})$ with respect to the standard basis $(\vec{e_1}\,; \vec{e_2})$.

c) Let $(\vec{v_1}\,; \vec{v_2})$ be another basis of $V_2$ defined by $\begin{cases} \vec{v_1} = 3\vec{e_1} + 2\vec{e_2} \\ \vec{v_2} = \vec{e_1} + \vec{e_2} \end{cases}$

Find the matrix $N$ of $T$ with respect to the basis $(\vec{v_1}\,; \vec{v_2})$.

d) Find the components of $\vec{v}$ with respect to the basis $(\vec{v_1}\,; \vec{v_2})$ and then use $N$ to find the components of $T(\vec{v})$ with respect to the basis $(\vec{v_1}\,; \vec{v_2})$.

e) Check that the answers b) and d) are "equal".

20. Let $V$ be the vector space of polynomials of degree $\leq 2$.

Consider the the linear transformation $T$ on $V$ given by

$T(ax^2 + bx + c) = 2ax^2 + bx - 2a$

a) Find the matrix of $T$ with respect to the basis $(1, x, x^2)$.
Then use this matrix to calculate the image of $p(x) = 4x^2 - 4x + 1$.

b) Find the matrix of $T$ with respect to the basis $(4x^2, x, 2)$.
Then use this matrix to calculate the image of $p(x) = 4x^2 - 4x + 1$.

21. Let $V$ be the vector space of polynomials of degree $\leq 2$.

A linear transformation $T$ on $V$ is given by its matrix $M = \begin{pmatrix} 3 & 0 & 1 \\ 1 & 3 & 3 \\ 0 & 1 & 0 \end{pmatrix}$

with respect to the basis $(x^2 + x\,, x - 1\,, 2)$.

Find the image under $T$ of $p(x) = 6x^2 + 10x - 6$

Give the answer in the form $ax^2 + bx + c$

## 7. Eigenvalues . Eigenvectors

## and

## 8. Diagonal matrix

22. For each of the following linear transformations, find

- the eigenvalues, the eigenvectors and the eigenspaces
- a basis consisting of eigenvectors and, if such a basis exists, the matrix of the linear transformation with respect to this basis.

A. In the plane $V_2$ with the standard basis $(\overrightarrow{e_1}\,;\,\overrightarrow{e_2})$

a) a projection onto the line parallel to $\vec{d} = \overrightarrow{e_1} + \overrightarrow{e_2}$

b) a reflection in the line $x + 3y = 0$

c) a dilatation with factor $k$

d) a rotation through 30°

B. In the space $V_3$ with the standard basis $(\overrightarrow{e_1}\,;\,\overrightarrow{e_2}\,;\,\overrightarrow{e_3})$

e) a projection onto the plane $x - 2y + z = 0$

f) a reflection in the plane $2x - y + 3z = 0$

g) a rotation through $60°$ about the line parallel to $\overrightarrow{e_1}$

h) a reflection in the line $d\colon \begin{cases} x = 3\lambda \\ y = \lambda \\ z = 2\lambda \end{cases}$

C. For $m = 0, 1$ and $2$, let $f_m\colon x \longmapsto y = e^{mx}$. Let $V$ be the vector space of all the linear combinations of $f_0$, $f_1$ and $f_2$. A basis of $V$ is $(f_0, f_1, f_2)$.

i) the differentiation $D\colon V \longrightarrow V\,, f \longmapsto f'$

j) the linear transformation $T$ defined by

$$T(\alpha_0 \cdot f_0 + \alpha_1 \cdot f_1 + \alpha_2 \cdot f_2) = \alpha_2 \cdot f_1 + \alpha_1 \cdot f_2$$

23. Let $T\colon V \longrightarrow V$ be a linear transformation and $\lambda \in \mathbb{R}$. Show that the set $W$ of all the vectors $\vec{v}$ such that $T(\vec{v}) = \lambda\vec{v}$ is a subspace of $V$.

24. Linear transformations on $V_n$ are given by their matrix with respect to the standard basis. For each of them find
   - the eigenvalues, the eigenvectors and the eigenspaces
   - a basis consisting of eigenvectors and, if such a basis exists, the matrix of the linear transformation with respect to this basis.

## 9. Matrix operations

25. Calculate matrix products and inverses of matrices.

## 10. Change of basis

26. In $V_2$ consider the linear transformation $T$ given by its matrix $M = \begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}$ with respect to the standard basis $(\overrightarrow{e_1}\,;\,\overrightarrow{e_2})$.

Let $(\overrightarrow{v_1}\,;\,\overrightarrow{v_2})$ be another basis of $V_2$ defined by $\begin{cases} \overrightarrow{v_1} = 3\overrightarrow{e_1} + 2\overrightarrow{e_2} \\ \overrightarrow{v_2} = \overrightarrow{e_1} + \overrightarrow{e_2} \end{cases}$

Find the matrix $M'$ of $T$ with respect to the basis $(\overrightarrow{v_1}\,;\,\overrightarrow{v_2})$.

27. Let $M = \begin{pmatrix} -10 & -9 \\ 12 & 11 \end{pmatrix}$. Calculate $M^{10}$.

28. Let $M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

a) Show the eigenvalues of $M$ are $\lambda_1 = \dfrac{1+\sqrt{5}}{2}$ and $\lambda_2 = \dfrac{1-\sqrt{5}}{2}$ and find the corresponding eigenvectors.

b) Diagonalize $M$ and show that $M^n = \dfrac{1}{\sqrt{5}}\begin{pmatrix} \dots & \lambda_1^n - \lambda_2^n \\ \dots & \dots \end{pmatrix}$

c) Consider the sequence of the Fibonacci numbers $u_n$ defined by

$u_0 = 0$ ; $u_1 = 1$ and $u_n = u_{n-1} + u_{n-2}$ for $n \geq 2$

Show by induction that $M^n = \begin{pmatrix} \dots & u_n \\ \dots & \dots \end{pmatrix}$

So $u_n = \dfrac{1}{\sqrt{5}}\left(\left(\dfrac{1+\sqrt{5}}{2}\right)^n - \left(\dfrac{1-\sqrt{5}}{2}\right)^n\right)$

## 11. Null space (kernel) and Range of a linear transformation

29. Find the null space and the range of each of the following linear transformations given with respect to the standard basis. For a) and b) find also the matrix of the transformation with respect to the standard bases.

a) $T: V_2 \to V_2, \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2x - y \\ x \end{pmatrix}$   b) $T: V_3 \to V_3, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x - y \\ z \\ 0 \end{pmatrix}$

c) The linear transformation $T$ on $V_3$ whose matrix with respect to the standard basis is $\begin{pmatrix} 1 & -3 & 5 \\ 4 & 0 & 8 \\ 2 & 1 & 3 \end{pmatrix}$

30. In the vector space $V$ of polynomials of degree $\leq 2$ consider the differentiation. Find the null space, the range and the matrix with respect to the basis $(1, x, x^2)$ of this transformation.

31. Let $V$ be the vector space of the functions $\mathbb{R} \to \mathbb{R}$ which are differentiable and $W$ be the vector space of all functions $\mathbb{R} \to \mathbb{R}$

Find the null space of the linear transformation $T: V \to W, f \mapsto f' - 3f$

32. Let $\vec{a}$ be a fixed non-zero vector. Find the null space of $T: V_3 \to \mathbb{R}, \vec{v} \mapsto \vec{a} \cdot \vec{v}$

## 13. Geometry of orthogonal transformations

33. Consider the transformation $T$ on $V_2$ given by $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \frac{1}{2}\begin{pmatrix} x + \sqrt{3}y \\ \sqrt{3}x - y \end{pmatrix}$ with respect to an orthonormal basis.

Show that $T$ is orthogonal and describe it geometrically.

34. In each case a linear transformation on $V_n$ is given by its matrix in an orthonormal basis. Show that the linear transformation is orthogonal and describe it geometrically.

a) $\frac{1}{13}\begin{pmatrix} 5 & 12 \\ -12 & 5 \end{pmatrix}$ b) $\frac{1}{7}\begin{pmatrix} 6 & -2 & 3 \\ -2 & 3 & 6 \\ -3 & -6 & 2 \end{pmatrix}$ c) $\frac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ 2 & 2 & -1 \\ -2 & 1 & -2 \end{pmatrix}$

35. In this exercise we use the standard basis of $V_3$.

a) Find the matrix $M_S$ of the reflection $S$ in the plane $x + y - 2z = 0$

b) Describe geometrically the linear transformation $T$ given by its matrix

$$M_T = \frac{1}{3}\begin{pmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix}$$

c) Find the matrix of the linear transformation $R$ defined by $R = T \circ S$ and geometrically describe $R$.

36. In the space $V_3$ an orthogonal transformation $T$ is given by its matrix

$$M = \frac{1}{7}\begin{pmatrix} 6 & -2 & 3 \\ -2 & 3 & 6 \\ -3 & -6 & 2 \end{pmatrix}$$

with respect to the standard basis.

Consider the plane $\pi: 2x - y + z = 0$ and its image $\pi'$ under $T$.

Find a Cartesian equation of the plane $\pi'$.

## 14. Affine transformations

37. Let $T$ be the linear transformation on $V_2$ given by its matrix $M = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$ with respect to the standard basis. Consider the affine transformation $f$ associated to $T$ and such that the image of the origin $O$ is the point $O'(-4;3)$.

    Let $d$ be the line through the points $A(2;3)$ and $B(5;1)$. Find a Cartesian equation of the image $d'$ of $d$ under $f$.

38. Let $T$ be the linear transformation on $V_3$ given by its matrix $M = \begin{pmatrix} 1 & 3 & -4 \\ -2 & 1 & 2 \\ 0 & -1 & 5 \end{pmatrix}$ with respect to the standard basis.

    Consider the affine transformation $f$ associated to $T$ and such that the image of the origin $O$ is the point $O'(1;2;-2)$.

    Consider the plane $\pi: 2x - y + z - 6 = 0$. Find a Cartesian equation of the image $\pi'$ of $\pi$ under $f$.

39. In $V_2$ consider the rotation $f$ through $120°$ which maps the point $A(-2;1)$ to the point $A'(1;0)$.

    Find the center of the rotation and the image of the origin $O$ under $f$.

**SOLUTIONS TO EXERCISES**

## 2. Basis. Components . Dimension

1. b) The vectors $\vec{v_1} = \vec{e_1} - \vec{e_2}$ and $\vec{v_2} = \vec{e_1} + \vec{e_2}$ are not parallel.

   c) $\vec{v} = 3\vec{v_1} + 5\vec{v_2}$ . Drawing.

2. $\vec{v_1} \wedge \vec{v_2} = 9\vec{e_1} + 28\vec{e_2} + 11\vec{e_3}$ ; $\vec{v} = 3\vec{v_1} - 2\vec{v_2} + \vec{v_1} \wedge \vec{v_2}$

3. Use Cramer's rule.

$$\begin{vmatrix} m & 2 & 1 \\ 2 & m & 2 \\ 1 & 2 & 1 \end{vmatrix} = m^2 - 5m + 4 = 0 \Rightarrow m = 1 \text{ or } m = 4 \text{ So, } m \neq 1 \text{ and } m \neq 4.$$

4. b) $\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 = p$, where $p$ is any polynomial

$$\Rightarrow \alpha_1(x^2 + 2x) + \alpha_2(x + 1) + \alpha_3(x^2 + 3) = ax^2 + bx + c$$

$$\Rightarrow \begin{cases} \alpha_2 + 3\alpha_3 = 5 \\ 2\alpha_1 + \alpha_2 = -4 \\ \alpha_1 + \alpha_3 = 3 \end{cases} ; \quad \begin{vmatrix} 0 & 1 & 3 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = -5 \neq 0, \text{ so } \alpha_1, \alpha_2 \text{ and } \alpha_3 \text{ exist and}$$

   are unique whatever the polynomial $p$.

   c) $p = 0p_1 - 4p_2 + 3p_3$

## 3, Subspaces

5. c) $x - y - z = 0$

6. For example $\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix}$

7. a) The vectors $\vec{v_1}$ and $\vec{v_2}$ belong to $W$ and they are not parallel.

   b) $m = -7$

   c) $\vec{v} = 3\vec{v_1} - 4\vec{v_2}$

   e) $\vec{v} = -1/2\,(2\vec{v_1}) - 4(\vec{v_2} - \vec{v_2})$

8. a) No, because for example $\vec{v_3} = -5/4\,\vec{v_1} - 3/2\,\vec{v_2}$

b) Basis: for example, $(\vec{v_1}\,;\,\vec{v_2})$; equation: $x - y + z = 0$

9. b) $p = ax^2 + bx + c = (x - m)(Ax + B) = Ax(x - m) + B(x - m)$

10. a) neither of the functions is a multiple of the other,

check it at $x = 0$ for example.

b) $g \notin V$ : if $g = \alpha s + \beta c$ then $x = 0 \Rightarrow 0 = \beta$ and $x = \pi/2 \Rightarrow 1 = \alpha$,

so $g = s$ but this is wrong.

$h(x) = \sin(x)\cos(\pi/4) - \cos(x)\sin(\pi/4)$

so $h = \sqrt{2}/2\, s - \sqrt{2}/2\, c$

**4. Linear transformations**

**5. Matrix for a linear transformation**

**and**

**6. Components of the image of a vector under a linear transformation**

13. d) $\frac{1}{13}\begin{pmatrix}4 & 6\\ 6 & 9\end{pmatrix}$ e) $\frac{1}{13}\begin{pmatrix}-5 & 12\\ 12 & 5\end{pmatrix}$ f) $\frac{1}{2}\begin{pmatrix}\sqrt{3} & -1\\ 1 & \sqrt{3}\end{pmatrix}$

g) $\begin{pmatrix}m & 0\\ 0 & m\end{pmatrix}$

h) $proj(\vec{v}) = \vec{p} = \vec{v} + m \cdot \vec{d}; \vec{p} \perp \vec{n} \Rightarrow m = -\dfrac{\vec{v}\cdot\vec{n}}{\vec{d}\cdot\vec{n}}$ ; $\dfrac{1}{5}\begin{pmatrix}2 & -6\\ -1 & 3\end{pmatrix}$

14. c) $\frac{1}{3}\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}$ e) $\frac{1}{3}\begin{pmatrix}2 & -1 & 1\\ -1 & 2 & 1\\ 1 & 1 & 2\end{pmatrix}$ f) $\frac{1}{3}\begin{pmatrix}1 & -2 & 2\\ -2 & 1 & 2\\ 2 & 2 & 1\end{pmatrix}$

g) $\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$

15. d) $\frac{1}{13}\begin{pmatrix}-10\\ -15\end{pmatrix}$ e) $\frac{1}{13}\begin{pmatrix}-46\\ 9\end{pmatrix}$ f) $\begin{pmatrix}\sqrt{3} + 3/2\\ 1 - 3\sqrt{3}/2\end{pmatrix}$

g) $\begin{pmatrix}2k\\ -3k\end{pmatrix}$ h) $\frac{1}{5}\begin{pmatrix}22\\ -11\end{pmatrix}$

16. c) $\frac{4}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}$ e) $\begin{pmatrix}4\\-1\\3\end{pmatrix}$ f) $\begin{pmatrix}6\\1\\1\end{pmatrix}$ g) $\begin{pmatrix}3\\2\\5\end{pmatrix}$

17. a) reflection in the $x$-axis b) reflection in the line $x - y = 0$
c) rotation through $-90°$ about O d) projection onto the $xz$-plane
e) reflection in $yz$-plane f) reflection in the plane $x - y = 0$
g) projection onto the $xz$-plane followed by a reflection in the $xy$-plane
h) rotation through $90°$ about the $x$-axis such that $\overrightarrow{e_2}$ goes to $\overrightarrow{e_3}$
i) reflection in the plane $y - z = 0$ followed by a dilatation from O with factor 2

18. a) $M = \begin{pmatrix}1 & 2 & 0\\0 & -2 & 1\end{pmatrix}$ b) $T(\vec{v}) = -5\overrightarrow{f_1} + 7\overrightarrow{f_2}$

19. a) $M = \begin{pmatrix}3 & 2\\-1 & 4\end{pmatrix}$

b) $T(\vec{v}) = \begin{pmatrix}4\\-6\end{pmatrix} = 4\overrightarrow{e_1} - 6\overrightarrow{e_2}$

c) $T(\overrightarrow{v_1}) = M \cdot \begin{pmatrix}3\\2\end{pmatrix} = \begin{pmatrix}13\\5\end{pmatrix} = 13\overrightarrow{e_1} + 5\overrightarrow{e_2} = 8\overrightarrow{v_1} - 11\overrightarrow{v_2}$

using $\begin{cases}\overrightarrow{e_1} = \overrightarrow{v_1} - 2\overrightarrow{v_2}\\ \overrightarrow{e_2} = -\overrightarrow{v_1} + 3\overrightarrow{v_2}\end{cases}$ or a system by means of the components with respect to the standard basis.

$T(\overrightarrow{v_2}) = M \cdot \begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}5\\3\end{pmatrix} = 5\overrightarrow{e_1} + 3\overrightarrow{e_2} = 2\overrightarrow{v_1} - 1\overrightarrow{v_2}$

$N = \begin{pmatrix}8 & 2\\-11 & -1\end{pmatrix}$

d) $\vec{v} = 2\overrightarrow{e_1} - \overrightarrow{e_2} = 3\overrightarrow{v_1} - 7\overrightarrow{v_2}$ ; $T(\vec{v}) = N \cdot \begin{pmatrix}3\\-7\end{pmatrix} = \begin{pmatrix}10\\-26\end{pmatrix} = 10\overrightarrow{v_1} - 26\overrightarrow{v_2}$

e) $T(\vec{v}) = 4\overrightarrow{e_1} - 6\overrightarrow{e_2} = 10\overrightarrow{v_1} - 26\overrightarrow{v_2}$

20. a) $\begin{pmatrix}0 & 0 & -2\\0 & 1 & 0\\0 & 0 & 2\end{pmatrix}$, $p = \begin{pmatrix}1\\-4\\4\end{pmatrix}$, $T(p) = \begin{pmatrix}-8\\-4\\8\end{pmatrix} = 8x^2 - 4x - 8$

b) $\begin{pmatrix}2 & 0 & 0\\0 & 1 & 0\\-4 & 0 & 0\end{pmatrix}$, $p = \begin{pmatrix}1\\-4\\1/2\end{pmatrix}$, $T(p) = \begin{pmatrix}2\\-4\\-4\end{pmatrix} = 8x^2 - 4x - 8$

21. $p = \begin{pmatrix} 6 \\ 4 \\ -1 \end{pmatrix}, T(p) = \begin{pmatrix} 17 \\ 15 \\ 4 \end{pmatrix} = 17x^2 + 32x - 7$

## 7. Eigenvalues . Eigenvectors

## and

## 8. Diagonal matrix

22. Part A in $V_2$

a) 1-eigenspace: the line $x - y = 0$; 0-eigenspace: the line $x + y = 0$

b) 1-eigenspace: the line $x + 3y = 0$; –1-eigenspace: the line $3x - y = 0$

c) all $V_2$ is the $k$-eigenspace

d) no eigenspace

Part B in $V_3$

e) 1-eigenspace: the plane $x - 2y + z = 0$; 0-eigenspace: the line parallel to $\vec{v} = \overrightarrow{e_1} - 2\overrightarrow{e_2} + \overrightarrow{e_3}$.

f) 1-eigenspace: the plane $2x - y + 3z = 0$ ; –1-eigenspace: the line parallel to $\vec{v} = 2\overrightarrow{e_1} - \overrightarrow{e_2} + 3\overrightarrow{e_3}$.

g) 1-eigenspace: the $x$-axis

h) 1-eigenspace: the line parallel to $\vec{v} = 3\overrightarrow{e_1} + \overrightarrow{e_2} + 2\overrightarrow{e_3}$,
–1-eigenspace: the plane $3x + y + 2z = 0$

Part C

*Remark*

A basis of $V$ is $(f_0, f_1, f_2)$ . It is easy to check.

$\alpha_0 f_0 + \alpha_1 f_1 + \alpha_2 f_2 = 0 \Leftrightarrow \alpha_0 f_0(x) + \alpha_1 f_1(x) + \alpha_2 f_2(x) = 0$ for all $x \in \mathbb{R}$

With $x = 0$ , $x = 1$, $x = 2$ you get the following system

$$\begin{cases} \alpha_0 + \alpha_1 + \alpha_2 = 0 \\ \alpha_0 + \alpha_1 e + \alpha_2 e^2 = 0 \\ \alpha_0 + \alpha_1 e^2 + \alpha_2 e^4 = 0 \end{cases}$$

Whose determinant is $-e + 2e^2 - 2e^4 + e^5 \neq 0$ . So $\alpha_0 = \alpha_1 = \alpha_2 = 0$

i) For each $m$ the function $f_m$ is an eigenvector corresponding to the eigenvalue $m$. So

0-eigenspace: all (scalar) multiples of $f_0$ ; 1-eigenspace: all (scalar) multiples of $f_1$ ; 2-eigenspace: all (scalar) multiples of $f_2$.

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

j) 0-eigenspace: all (scalar) multiples of $f_0$ ; 1-eigenspace: all (scalar) multiples of $f_1 + f_2$ ; -1-eigenspace: all (scalar) multiples of $f_1 - f_2$

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

With respect to the basis $(f_0, f_1 + f_2, f_1 - f_2)$.

23. *Remark*

For most of the values of $\lambda$ the subset $W$ contains only the zero vector.

## 9. Matrix operations

## 10. Change of basis

26. $M' = P^{-1}MP = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 2 \\ -11 & -1 \end{pmatrix}$

27. $M = PDP^{-1}$ with $P = \begin{pmatrix} 3 & 2 \\ -4 & -2 \end{pmatrix}, D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$ and $P^{-1} = \frac{1}{2}\begin{pmatrix} -2 & -2 \\ 4 & 3 \end{pmatrix}$

$M^{10} = PD^{10}P^{-1} = \begin{pmatrix} -3068 & -3069 \\ 4092 & 4093 \end{pmatrix}$

28. a) $\det(M-\lambda I)=\lambda^2-\lambda-1=0 \Rightarrow \lambda=\frac{1\pm\sqrt{5}}{2}$

$Remark\ \lambda_1+\lambda_2=-\lambda_1\lambda_2=1$

an eigenvector for $\lambda_1=\frac{1+\sqrt{5}}{2}$ is $\begin{pmatrix}1\\-\lambda_2\end{pmatrix}$

an eigenvector for $\lambda_2=\frac{1-\sqrt{5}}{2}$ is $\begin{pmatrix}1\\-\lambda_1\end{pmatrix}$

b) $M^n=PD^nP^{-1}=\begin{pmatrix}1&1\\-\lambda_2&-\lambda_1\end{pmatrix}\begin{pmatrix}\lambda_1^n&0\\0&\lambda_2^n\end{pmatrix}\frac{-1}{\sqrt{5}}\begin{pmatrix}-\lambda_1&-1\\\lambda_2&1\end{pmatrix}$

c) $M^n=\begin{pmatrix}u_{n+1}&u_n\\u_n&u_{n-1}\end{pmatrix}$ for $n\geq 1$

## 11. Null space (kernel) and range

29. a) $N(T)=0\,;R(T)=V_2\,;\ M=\begin{pmatrix}2&-1\\1&0\end{pmatrix}$

b) $N(T)$ is the line parallel to $\begin{pmatrix}1\\1\\0\end{pmatrix}$, $R(T)$ is the plane $z=0$

$$M=\begin{pmatrix}1&-1&0\\0&0&1\\0&0&0\end{pmatrix}$$

c) $N(T)$ is the line parallel to $\begin{pmatrix}2\\-1\\-1\end{pmatrix}$, $R(T)$ is the plane normal to $T(\vec{e_1})\wedge T(\vec{e_2})=\begin{pmatrix}4\\-7\\12\end{pmatrix}$, so the plane $4x-7y+12z=0$

## 13. Geometry of orthogonal transformations

33. Reflection in the line with equation $x - \sqrt{3}y = 0$

34. a) Rotation through (by) $-67.38\ldots°$

b) Rotation about the line parallel to $\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ through $73.39\ldots°$

c) Reflection in the plane $x + 2z = 0$ followed by a rotation about the line parallel to $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ through $48.18\ldots°$

35. a) $M_S = \frac{1}{3}\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & -1 \end{pmatrix}$

b) Reflection in the plane with equation $x - 2y + z = 0$

c) $M_R = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

Rotation about the line parallel to $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ through $120°$

*Remark* The intersection of the fixed planes of $S$ and $T$ is the axis of the rotation and the acute angle between these planes is $60° = \frac{1}{2} \cdot 120°$, that is half of the angle of the rotation.

36. Choose a basis of $\pi$.

$$\vec{v_1} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \perp \vec{n} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}; \vec{v_2} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \perp \vec{n} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$T(\vec{v_1}) = \frac{1}{7}\begin{pmatrix} 2 \\ 4 \\ -15 \end{pmatrix} \parallel \begin{pmatrix} 2 \\ 4 \\ -15 \end{pmatrix}; T(\vec{v_2}) = \frac{1}{7}\begin{pmatrix} 0 \\ -14 \\ -7 \end{pmatrix} \parallel \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

So $\pi'$ is the subspace of all the linear combinations of $T(\vec{v_1})$ and $T(\vec{v_2})$.

Each linear combination is normal to

$$T(\overrightarrow{v_1}) \wedge T(\overrightarrow{v_1}) = \begin{pmatrix} 2 \\ 4 \\ -15 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 34 \\ -2 \\ 4 \end{pmatrix} \parallel \begin{pmatrix} 17 \\ -1 \\ 2 \end{pmatrix}$$

So $\pi': 17x - y + 2z = 0$

*Other method*

Since $T$ is orthogonal $\vec{n} \cdot \vec{v} = 0$ implies $T(\vec{n}) \cdot T(\vec{v}) = 0$. So $T(\vec{n})$ is the vector normal to the plane $\pi'$. Then $\pi': 17x - y + 2z = 0$.

## 14. Affine transformations

37. $d: \overrightarrow{OP} = \overrightarrow{OA} + \lambda \cdot \overrightarrow{AB}$

$$d': \overrightarrow{OP'} = T(\overrightarrow{OP}) + \overrightarrow{OO'} = T(\overrightarrow{OA} + \lambda\overrightarrow{AB}) + \overrightarrow{OO'} =$$

$$= T(\overrightarrow{OA}) + \lambda T(\overrightarrow{AB}) + \overrightarrow{OO'} = \overrightarrow{OA'} + \lambda T(\overrightarrow{AB})$$

$$d': \overrightarrow{OP'} = \begin{pmatrix} -4 \\ 7 \end{pmatrix} + \begin{pmatrix} -4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ -9 \end{pmatrix} = \begin{pmatrix} -8 \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ -9 \end{pmatrix}, \quad d': 9x + 7y + 2 = 0$$

38. Choose three points in $\pi$: $A(3; 0; 0), B(0; -6; 0)$ and $C(0; 0; 6)$ for example.

$\pi: \overrightarrow{OP} = \overrightarrow{OA} + \lambda\overrightarrow{AB} + \mu\overrightarrow{AC}$ with and

$$\pi': \overrightarrow{OP'} = T(\overrightarrow{OP}) + \overrightarrow{OO'} = T(\overrightarrow{OA}) + \lambda T(\overrightarrow{AB}) + \mu T(\overrightarrow{AC}) + \overrightarrow{OO'} =$$

$$= \begin{pmatrix} 3 \\ -6 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -21 \\ 0 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} -27 \\ 18 \\ 30 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -21 \\ 0 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} -27 \\ 18 \\ 30 \end{pmatrix}$$

$$\begin{pmatrix} -7 \\ 0 \\ 2 \end{pmatrix} \wedge \begin{pmatrix} -9 \\ 6 \\ 10 \end{pmatrix} = \begin{pmatrix} -12 \\ 52 \\ -42 \end{pmatrix} \parallel \begin{pmatrix} 6 \\ -26 \\ 21 \end{pmatrix} \perp \pi'$$

$\pi': 6x - 26y + 21z - 86 = 0$

39. $\begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$ where the image of the origin is $(t_1; t_2)$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix} + \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \Rightarrow t_1 = \frac{\sqrt{3}}{2}; t_2 = \frac{1}{2} + \sqrt{3}$$

The center of the rotation is the fixed point

$$\left(\frac{1-\sqrt{3}}{2\sqrt{3}}; \frac{1+\sqrt{3}}{2}\right) = (-0.2113 \ldots; 1.3660 \ldots)$$