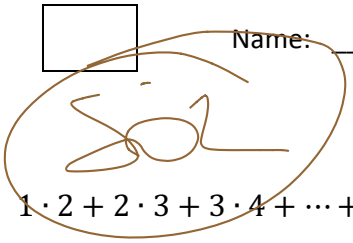
3MG Level 2	INDUCTION COMBINATORICS	TEST#1A
2018/10/02	PROBABILITIES CALCULUS	3MG01

With formulaire /42 pts Name: _____ 90'



Exercise 1. [/8pts]

1) Use induction to show that for $n \geq 1$: $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}$

2) Verify the inductive step for the assertion " $4^n + 1$ is divisible by 3 for $n \in \mathbb{N}$ " and determine whether the assertion is true or not.

Ex 1 2) Step: $4^n+1 \in M_3 \Rightarrow 4^{n+1}+1 \in M_3$?

$4^n+1 \in M_3$ so $4^n+1 = 3m, m \in \mathbb{N}$ so $4^n = 3m-1$

$4^{n+1}+1 = 4 \cdot 4^n + 1 = 4 \cdot (3m-1) + 1 = 12m - 4 + 1 = 12m - 3 = 3 \cdot (4m-1) \in M_3$ ✓

Conclusion: Check the basis! For $n=0$ $4^0+1 = 2 \notin M_3 \Rightarrow$ Statement wrong!

Exercise 2. [/8pts]

1) How many different anagrams of the word « *SENTENCE* » are there ? STC, 3E, 2N (8 letters) $\frac{8!}{3! \cdot 2!} = 3360$

2) How many different ways are there to select two letters from « *SENTENCE* » ? EE, NN, SENTC $C_2^5 = \frac{5 \cdot 4}{2}$ $2 + 10 = 12$

3) When forming at random an anagram of the word « *SENTENCE* » what's the probability that the three "*E*" are "together" ? EEE SNTNC (6 "letters") $\frac{6!}{2!} = 360$ So $p = \frac{360}{3360} = \frac{3}{28}$

4) A bunch of 10 roses is to be formed from 6 different possible colors. How many different such bunches are there ? (*the order of the flower not being important, only the colors matter*)

5) Determine the number $\frac{100!}{96! \cdot 3!} = \frac{50 \cdot 33 \cdot 100 \cdot 99 \cdot 98 \cdot 97}{3 \cdot 2} = 156849900$

4) C_1 C_2 C_3 ... C_6

x x | x | x x | .. | x $10 + 5 = 15$ $\frac{15!}{10! \cdot 5!} = 3003$

Ex 1 1) Basis True for $n=1$? $\frac{1 \cdot 2 \cdot 3}{3} = 2 \stackrel{?}{=} 1 \cdot 2$ ✓

Step Hyp : True for n

Step : True for $n+1$? $S_{n+1} \stackrel{?}{=} (n+1)(n+2)(n+3)/3$

$\underbrace{1 \cdot 2 + \ldots + n(n+1)}_{S} + (n+1)(n+2) \overset{(H)}{=} \frac{n(n+1)(n+2)}{3} + \frac{(n+1)(n+2) \cdot 3}{3} = \frac{(n+1)(n+2) \cdot (n+3)}{3}$ ✓

Exercise 3. [/16pts]

1) In a sample space we consider two events A and B such that $p(A) = 0.75, P(\bar{B}) = 0.4$ and $p(A \cap B) = 0.4$. Determine $p(A \cup B)$, $p(\bar{A} \cap B)$ and $p(A|\bar{B})$.

2) Someone invites you to play that game

"Let's roll two fair six-sided dice. If the product of the points is (strictly) smaller than 6 points, I win 10.- ; if the sum is (strictly) larger than 10 points I win 20.- Else you win x.-"

For what possible amounts *x* (in swiss francs) do you accept to play ? Justify your answer.

3) When selecting 2 cards from a 36 cards deck. What's the probability that
- the cards have different colors ? (*colors : hearth, diamond, club, spade*)
- the cards have same symbol or same color ? (*symbols : 6,7,8,9,10, jack, queen, king, ace*)

4) Two types of confetti bags have been sold last week-end.

60% of them were of "Type A" that contains 20% of red, 30% of yellow and 50% of blue confettis.

The other were of "Type B" that contains 35% of red, 50% of yellow and 15% of green confettis.

- Draw the tree diagram of the situation.
- Determine the probability for a randomly found piece of confetti to be red.
- A randomly found piece of confetti is green. What's its probability to come from a "Type A" bag ?
- A randomly found piece of confetti is red. What's its probability to come from a "Type B" bag ?

Ex 3.1)

$p(B) = 0,6$

$p(A \cup B) = 0,95$

$p(\bar{A} \cap B) = 0,2$

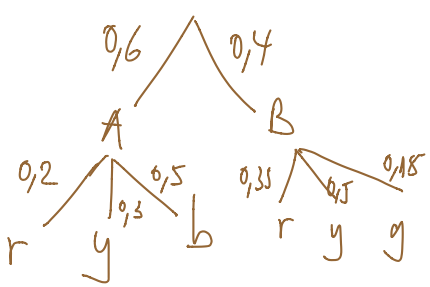
$p(A|\bar{B}) = \frac{0,35}{0,40} = \frac{35}{40} = \frac{7}{8}$

Ex 3.2) $p(\text{prod} < 6) = 10/36$ $p(\text{sum} > 10) = 3/36$ $p(\text{other}) = 23/36$

$E = \frac{10}{36} \cdot (10.-) + \frac{3}{36} \cdot (20.-) - x \cdot \frac{23}{36} \geq 0$ $x \geq \frac{160}{23} \cong 6,957$ so $x \geq 7.-$

Ex 3.3) $27/35$ and $8/35 + 3/35 = 11/35$

Ex 3.4)



$p(r) = 0,6 \cdot 0,2 + 0,4 \cdot 0,35 = \boxed{0,26}$

$P(A|g) = \boxed{0}$

$P(B|r) = \frac{p(B \cap r)}{p(r)} = \frac{0,4 \cdot 0,35}{0,26} = \frac{0,14}{0,26} = \boxed{\frac{7}{13}}$

Exercise 4. [/10pts]

1) What is the geometrical meaning of the number $f'(a)$ for a given function f and a real number a ? Give a precise answer.

2) Determine the derivative of $f(x) = \cos(3x) + \frac{4x^2-x}{2x+1}$

3) Determine the equation of the tangent to the graph of $f(x) = x^2 \cdot \sin(x)$ at its point with abscissa $x = 2$.

4) We consider the graph of the function $f(x) = x^4 - 8x + 10$. Determine the coordinates of the point on that curve that is the closest to the line $y = 2$. What's the measure of that shortest distance ?

1) Slope of the line tangent to the graph of f at its point with abscissa $x=a$.

2) $(\cos(3x))' = -3\sin(3x)$ $\left(\frac{4x^2-x}{2x+1}\right)' = \frac{(8x-1)(2x+1)-(4x^2-x)\cdot 2}{(2x+1)^2}$

$= \frac{16x^2+6x-1-8x^2+2x}{(2x+1)^2} = \frac{8x^2+8x-1}{(2x+1)^2}$

So $f'(x) = -3\sin(3x) + \frac{8x^2+8x-1}{(2x+1)^2}$

3) $f'(x) = 2x\cdot\sin(x) + x^2\cdot\cos(x)$

At $x=2$ ⚠ rad $f(2) \simeq 3,64$ $f'(2) \simeq 1,97$

$t: y = 1,97\cdot x + h$

$3,64 = 1,97\cdot 2 + h$

$\Rightarrow h \simeq -0,31$

$t: y \simeq 1,97x - 0,31$

4) $P(x;y) \in \mathcal{C}_f$ dist to line $y=2$ is $d(x) = |f(x)-2|$

So $d(x) = |x^4 - 8x + 8|$

Optimisation ! $d'(x) = 0$

$d'(x) = 4x^3 - 8 = 0$

$x^3 = 2$

$x = \sqrt[3]{2} \simeq 1,26$

$y = f(1,26) \simeq 2,44$ $P(1,26; 2,44)$

1.1 neglected

dist $= 2,44 - 2 \simeq 0,44$