LDDR – Niveau II : Calcul Integral

1. 24 :

1. 25 : Find antiderivatives of the following :

1) $f(x) = x^4 - 6x^2 + 5$
2) $f(x) = \dfrac{4}{x^2} - 1$
3) $f(x) = 3\sin(x)$
4) $f(x) = \sin(3x)$
5) $f(x) = x^3 + \dfrac{1}{x^3}$
6) $f(x) = (x+1)^3$
7) $f(x) = (2x-3)^5$
8) $f(x) = \dfrac{1}{(x+3)^2}$
9) $f(x) = \dfrac{1}{\cos^2(x)}$
10) $f(x) = \tan^2(x)$
11) $f(x) = 1 + \tan^2(2x)$
12) $f(x) = \dfrac{1}{\sqrt{x+1}}$
13) $f(x) = \dfrac{-4}{x^4} + \dfrac{5}{x^5}$
14) $f(x) = \dfrac{2x^3 + x^2 - 1}{x^2}$

1. 26 : Evaluate the following definite integrals :

1) $\displaystyle\int_2^5 x^2\,dx$
2) $\displaystyle\int_1^2 \frac{4}{3}x^3\,dx$
3) $\displaystyle\int_0^2 \left(x^3 + 2x + 1\right)\,dx$
4) $\displaystyle\int_{-1}^2 \left(2x^3 - x^2 + 3x - 4\right)\,dx$
5) $\displaystyle\int_{-1}^2 \left(x^4 - 2x^3 + 3x^2 - 2.7\right)\,dx$
6) $\displaystyle\int_1^4 \frac{1}{x^2}\,dx$
7) $\displaystyle\int_0^{\pi} \sin(x)\,dx$
8) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos(x)\,dx$
9) $\displaystyle\int_1^3 \left(x + \frac{1}{x}\right)^2\,dx$
10) $\displaystyle\int_{-1}^1 \frac{1}{x^2+1}\,dx$

1. 27 : Compute the following integrals :

1) $\displaystyle\int_{-1}^1 e^{-x}\,dx$
2) $\displaystyle\int_{-3}^9 e^{x\ln(3)}\,dx$
3) $\displaystyle\int_{\ln(3)}^{\ln(9)} e^x\,dx$
4) $\displaystyle\int_0^1 a^x\,dx$
5) $\displaystyle\int_{-3}^{-1} \frac{1}{x}\,dx$
6) $\displaystyle\int_1^3 \frac{1}{x-5}\,dx$
7) $\displaystyle\int_0^8 \sqrt[3]{x}\,dx$
8) $\displaystyle\int_1^e \ln(x^2)\,dx$

1.28 : Determine the average value $\bar{f}$ of f on the given interval. Then, determine a number c in the interval such that $f(c) = \bar{f}$.

1) $f(x) = \dfrac{1}{\sqrt{x}}$ on $I = [1; 9]$
2) $f(x) = \dfrac{1}{\sqrt{x}}$ on $I = [4; 9]$
3) $f(x) = \cos(x)$ on $I = [0; \pi]$
4) $f(x) = \cos(x)$ on $I = \left[\frac{\pi}{2}; \pi\right]$
5) $f(x) = \dfrac{1}{x^2}$, $I = [a; b]$, with $b > a > 0$

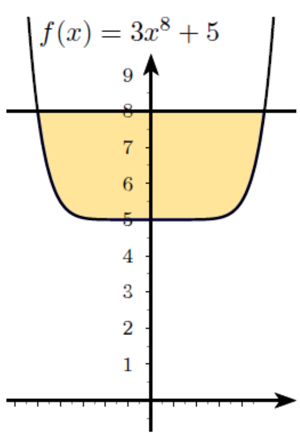
1.29 : We consider the function $f(x) = e^{0,75x} - 3x + c$.

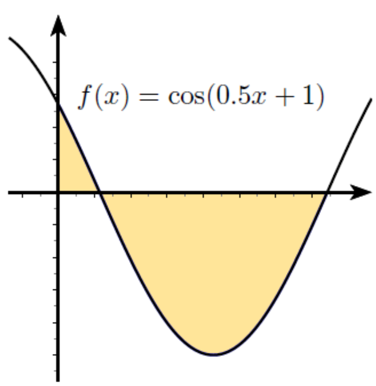
1) Determine the value of c so that f passes through the origin.
2) Compute the coordinates of the stationary point (with the value of c obtained at 1)).
3) Analyse the asymptotic behaviour of f and plot it.
4) Determine the value of $\int_{-1}^{0} f(x)dx$

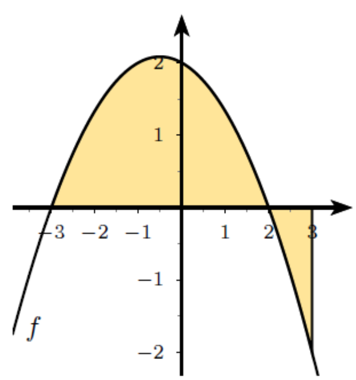
1.30 : Answer the following :

1) Calculate the area under the graph of $f(x) = x^4 + 5$ from $x = -1$ to $x = 1$.
2) Determine the area between the x-axis and the curve $f(x) = x^2 - 5$ from $x = -3$ to $x = 2$.

1.31 : In each case, determine the area of the shaded region :







1.32 : In each situation, plot the graphs of f and g on the same coordinate system, hatch the surface delimited by the two curves and determine its area.

1) $f(x) = \frac{1}{2}x^2$ and $g(x) = 4 - x$

2) $f(x) = 1 + \frac{1}{9}x^2$ and $g(x) = 3 - \frac{1}{9}x^2$

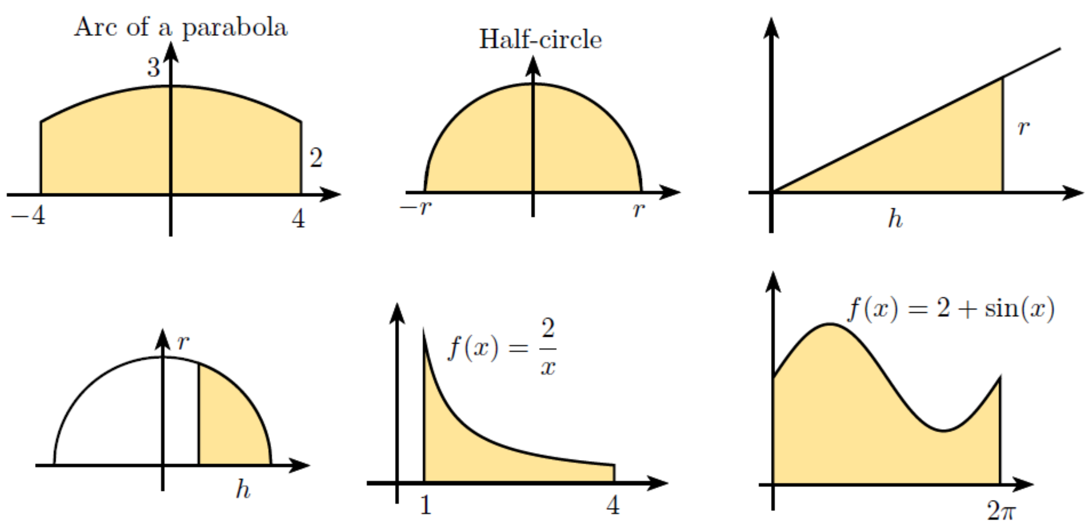
3) $f(x) = -3x^3 + 9x^2$ and $g(x) = -3x^2 + 12x$

4) $f(x) = \frac{4}{x^2}$ and $g(x) = \frac{17 - x^2}{4}$

1.33 :

1.34 : In the first quadrant we consider the area bounded by the axes of coordinates, the line $y = 8$ and the graph of the function $f(x) = \frac{4 - x^2}{x^2} = \frac{4}{x^2} - 1$. Represent that graph. Find the height of the horizontal line that divides this area in two equal parts.

1.35 : Determine the volume of the following solid of revolution obtained by rotating the shaded areas about the x-axis.



1.36 : We consider two functions : $f(x) = \frac{x^2}{a}$ and $g(x) = \frac{a}{x^2}$.

1) Find the positive value of a such that $f(x)$ and $g(x)$ intersect themselves with a right angle.

2) With $a = 4$ calculate the area between the two curves, the x-axis and the vertical line $x = 4$.

3) Calculate the volume of revolution generated by the rotation about the x-axis of the area defined at 2).

1.37 : Find the volume generated when the region bounded by the graph of $y = f(x)$, the y-axis and the lines $y = c$ and $y = d$ is rotated about the y-axis to form a solid of revolution.

1) $f(x) = x^2 + 1,\ \ c = 1$ and $d = 4$

2) $f(x) = x^{2/3},\ \ c = 1$ and $d = 5$

1.38 : Draw the functions $f(x) = \frac{4}{x}$ and $g(x) = 5x - x^3$ in the same set of axes and hatch the area, in the first quadrant, bounded by the two graphs.
Determine the volume of the ring obtained by rotating the hatched area about the x-axis.

1.39 : Determine the length L of the arc on the curve $f(x) = \frac{e^x + e^{-x}}{2}$ on the interval $[0; 5]$.

1.40 : Determine the antiderivatives of the following functions, by using the integration by parts :

1) $f(x) = x \cdot a^x$

2) $f(x) = \ln(x)$

3) $f(x) = e^x \cdot \sin(x)$

4) $f(x) = (x^2 - x + 1)\, e^x$

5) $f(x) = x^2 \sin(x)$

6) $f(x) = x(1 - x)^4$

7) $f(x) = x \ln(x)$

8) $f(x) = x e^x \cos(x)$

9) $f(x) = x\sqrt{x + 1}$

1.41 : Compute $\displaystyle\int_0^{\pi} \cos^2(x)\, dx$.

1.42 : In each case, analyse f and g in order to plot them on the same coordinate system. Hatch and compute the area bounded by the two curves.

1) $f(x) = x^2 e^{-x}$ and $g(x) = e^{-x}$
2) $f(x) = \ln(x)$ and $g(x) = \ln^2(x)$
3) $f(x) = x^2 + 2x - 3$ and $g(x) = (x^2 + 2x - 3)e^x$

1.43 : Use integration by substitution to find an antiderivative for each of the following :

1) $f(x) = \cos(ax + b)$
2) $f(x) = (2x + 5)^4$
3) $f(x) = (ax + b)^n$
4) $f(x) = x(1 - x^2)^4$
5) $f(x) = \sin(x)\cos(x)$
6) $f(x) = x\cos(\pi x^2)$
7) $f(x) = x^3\sqrt{1 + x^4}$
8) $f(x) = x^2 e^{-x^3}$
9) $f(x) = \dfrac{\ln(x)}{x}$
10) $f(x) = \dfrac{e^x}{1 + e^x}$
11) $f(x) = \dfrac{2x + 1}{x^2 + x + 4}$
12) $f(x) = (x^2 - x)^4(2x - 1)$
13) $f(x) = \dfrac{x}{\sqrt[3]{3x^2 - 2}}$
14) $f(x) = \dfrac{x}{\sqrt[3]{3x - 2}}$
15) $f(x) = \dfrac{1}{\sqrt{1 - 9x^2}}$

1.44 : Use you « FAT » to propose an antiderivative for each of the following :

1) $f(x) = \dfrac{1}{x^2 + 4x - 5}$
2) $f(x) = \dfrac{6}{9x^2 - 4}$
3) $f(x) = \dfrac{1}{4x^2 - 3x}$
4) $f(x) = \dfrac{1}{x^2 - 12x + 36}$
5) $f(x) = \dfrac{2}{16x^2 - 8x + 1}$
6) $f(x) = \dfrac{x + 1}{(3x + 2)^2}$
7) $f(x) = \dfrac{1}{x^2 + 4x + 6}$
8) $f(x) = \dfrac{x^4 + x - 3}{x^2}$
9) $f(x) = \dfrac{1}{x^2 + 2x + 2}$
10) $f(x) = \dfrac{1}{x^2 + 4}$
11) $f(x) = \dfrac{x^2 - 3x}{x + 1}$
12) $f(x) = \dfrac{2x^4 - 4}{1 + 2x^2}$
13) $f(x) = \dfrac{x}{x^2 + 4x + 5}$
14) $f(x) = \dfrac{1}{x(x - 1)(x - 2)}$
15) $f(x) = \dfrac{x}{x^4 + 9}$

1.45 : We denote by I_n the value of the integral $I_n = \int_0^1 x^n\sqrt{1 - x}\,dx$.

1) Compute I_0 and I_1
2) Show that $(2n + 3) \cdot I_n = 2n \cdot I_{n-1}$
3) Compute I_2 and I_3
4) Express I_n as a function of n

1. 46 : Determine the area of the surface delimited by the graph of f, the x-axis and the vertical through the maximal value of f with :

1) $f(x) = x\mathrm{e}^{-x}$
2) $f(x) = \dfrac{24x}{(x^2+3)^2}$

1. 47 : For $n \in \mathbb{N}$, we consider f_n the function defined by $f_n(x) = x^n \cdot \mathrm{e}^{-x^2}$, F_n the antiderivative of f_n, that is $F_n(x) = \int_0^x t^n \cdot \mathrm{e}^{-t^2} dt$ and I_n the limit of $F_n(x)$ when x tends to infinity.

1) Compute $F_1(x)$ and I_1.
2) By using integration by parts to determine F_n, prove the result :

$$(n+1) \cdot F_n(x) = f_{n+1}(x) + 2 \cdot F_{n+2}(x)$$

3) Considering a limit, deduce that $I_{n+2} = \frac{1}{2}(n+1)\, I_n$ from the previous result.
4) Given that $I_0 = \frac{\sqrt{\pi}}{2}$, compute I_2, I_4, I_6, ...

1. 48 : Answer the following :

1) Determine the values of the improper integrals $\displaystyle\int_0^{16} \frac{1}{\sqrt[4]{x}}\, dx$ and $\displaystyle\int_0^{16} \frac{1}{\sqrt[4]{x^3}}\, dx$.
2) Determine the values of the infinite integrals $\displaystyle\int_2^{+\infty} \frac{6}{x^4}\, dx$ and $\displaystyle\int_4^{+\infty} \frac{6}{x\sqrt{x}}\, dx$.
3) Determine, as a function of m and s, the value of $\displaystyle\int_1^{s} \frac{1}{x^m}\, dx$, with $m \in \mathbb{Q}_+ \backslash \{1\}$ and $s > 0$.
4) Indicate for which values of m the expression $\displaystyle\int_1^{+\infty} \frac{1}{x^m}\, dx$ is computable.

1.49 : Let's consider the functions $f(x) = \frac{x^3}{x^2-3x+2}$, $g(x) = \frac{x^2}{x^2+4x+4}$ and $h(x) = \frac{x}{2x^2+x+1}$.

1) Indicate their domain.

2) Compute, whenever possible, the integral of f from a to b with :

   a) $a = 1,5$ and $b = 2$
   b) $a = -3$ and $b = 0$
   c) $a = -1$ and $b = 1$

3) Compute, whenever possible, the integral of g from a to b with :

   a) $a = 0$ and $b = 2$
   b) $a = -2$ and $b = 0$
   c) $a = -3$ and $b = 0$

4) Compute, whenever possible, the integral of h from a to b with :

   a) $a = -0,25$ and $b = 0$
   b) $a = 0$ and $b = +\infty$

✏

1.50 : Determine, whenever possible, the value of the following infinite integrals :

1) $\displaystyle\int_2^{\infty} \frac{1}{2x^2 - x - 1}\,dx$

2) $\displaystyle\int_0^{2} \frac{1}{2x^2 - x - 1}\,dx$

3) $\displaystyle\int_1^{\infty} \frac{x+3}{(x+1)^3}\,dx$

4) $\displaystyle\int_{-1}^{1} \frac{3x^2}{(1+2x^3)^2}\,dx$

5) $\displaystyle\int_{1,5}^{\infty} \frac{1}{2x^2 - 6x + 5}\,dx$

6) $\displaystyle\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx$

✏

1.51 : Determine the exact values and the estimations of Simpson of :

1) $\displaystyle\int_a^b x^2\,dx$

2) $\displaystyle\int_a^b x^3\,dx$

3) $\displaystyle\int_0^{\pi/3} \sin(x)\,dx$

4) $\displaystyle\int_0^{1} \frac{4}{1+x^2}\,dx$

Determine the relative error, in %, of each estimation.

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1.52 : We consider the function $f(x) = \mathrm{e}^{-\frac{1}{2}x^2}$, the normal distribution used in statistics. It's also called the curve of Gauss.

1) Analyse the function f and plot its graph.

2) Compute the estimations of Simpson of $I_1 = \int_0^1 f(x)\,dx$, $I_2 = \int_1^2 f(x)\,dx$, $I_3 = \int_2^3 f(x)\,dx$, $I_4 = \int_3^4 f(x)\,dx$, $I_5 = \int_4^5 f(x)\,dx$.

3) Deduce an estimation of $\int_{-5}^{5} f(x)\,dx$ and of $\int_{-\infty}^{\infty} f(x)\,dx$.

1.53 : Given the function $f(x) = \begin{cases} \dfrac{\sin(x)}{x} & \text{if } x \neq 0 \\ a & \text{if } x = 0 \end{cases}$

1) Determine a for the function to be continuous.

2) Determine its zeros, plot the hyperbolas $y = \frac{1}{x}$ and $y = \frac{-1}{x}$, and draw f. Is it continuous everywhere ? Chose 2 squares as the unit on Ox and 10 on Oy.

3) Determine $\int_a^b f(x)\,dx$, using the formula of Simpson, in the following cases :

   a) $a = 0$, $b = \pi$

   b) $a = \pi$, $b = 2\pi$

   c) $a = n\pi$, $b = (n+1)\pi$, $n \in \mathbb{N}$

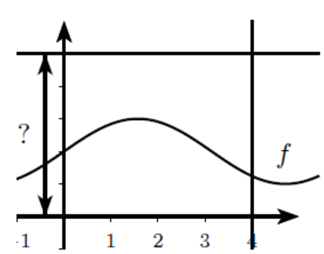
4) Is the integral $\int_0^\infty f(x)\,dx$ defined ?

1.54 : Answer the following :

1) Analyse the function $f(x) = \mathrm{e}^{\frac{1}{x}}$.

2) Estimate the integral of f from -2 to 0.

1.55 : What must the height of that rectangular target be so that a dart thrown randomly has the same probability to hit the target above and below the curve $f(x) = \sin(x) + 2$?



1.56 : The integral $\int_0^4 \frac{\sin(x)}{x} + 1\,dx$ can't be computed because there are no antiderivatives. How can you use the information that out of 1500 darts randomly thrown in the rectangle, 780 of them have hit the upper part ?