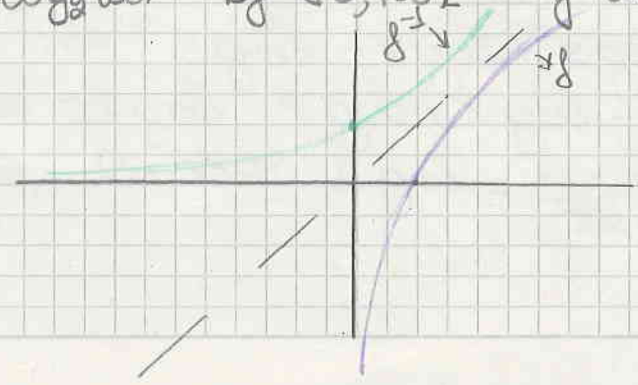
LJP : FONCTION EXPONENTIELLES-LOGARITHMIQUES

EXERCICE 1

1) $\log_3(81) = \log_3 3^4 = 4$
2) $\log_{13}(1) = 0$
3) $\log_5(125) = \log_5 5^3 = 3$
4) $\log(10) = 1$
5) $\log_4(8) = \frac{\log_2 8}{\log_2 4} = \frac{\log_2 2^3}{\log_2 2^2} = \frac{3}{2}$
6) $\log_a\left(\frac{1}{a}\right) = \log_a a^{-1} = -1$
7) $\log_9(27) = \frac{\log_3 27}{\log_3 9} = \frac{\log_3 3^3}{\log_3 3^2} = \frac{3}{2}$
8) $\log_{2/3}\left(\frac{16}{81}\right) = \log_{2/3}\left(\frac{2}{3}\right)^4 = 4$
9) $\log_2\left(\frac{1}{16}\right) = \log_2 2^{-4} = -4$
10) $\log(0.001) = \log 10^{-3} = -3$
11) $\log_a(1) = 0$
12) $\log_{1/a}(a) = \frac{\log a}{\log a^{-1}} = \frac{\log a}{-\log a} = -1$
13) $\log_8(2) = \frac{\log_2 2}{\log_2 2^3} = \frac{1}{3}$
14) $\log_5\left(\frac{1}{625}\right) = \log_5 5^{-4} = -4$
15) $\log_2(0.125) = \log_2 2^{-3} = -3$
16) $\log_{1/2}(32) = \frac{\log_2 2^5}{\log_2 2^{-1}} = \frac{5}{-1} = -5$
17) $\log_a(a) = 1$
18) $\log_a \sqrt[n]{a} = \log_a a^{1/n} = \frac{1}{n}$

EXERCICE 2

$f(x) = \log_2(x) \quad D_f =]0, +\infty[\quad f^{-1}(x) = 2^x$



EXERCICE 3

1) $x = \log_2 3 = \frac{\log 3}{\log 2} \approx 1.58$

2) $5 \cdot 3^x = 20 \Leftrightarrow 3^x = 4 \Rightarrow \log 3^x = \log 4 \Rightarrow x \log 3 = \log 4 \Rightarrow$
$x = \frac{\log 4}{\log 3} \approx 1.262$

3) $3^x = 2^{x+1} \Leftrightarrow x \log 3 = (x+1)\log 2 \Rightarrow x\log 3 = x\log 2 + \log 2$
$\Leftrightarrow x(\log 3 - \log 2) = \log 2 \Leftrightarrow x = \frac{\log 2}{\log 3 - \log 2} = \frac{\log 2}{\log \frac{3}{2}} \approx 1.71$

4) $\log_4 (x-1) = 2 \Leftrightarrow$ $\qquad x-1>0 \Leftrightarrow x>1$

$4^2 = x-1 \Leftrightarrow x = 17$ ✓

5) $\log_5 (x^2) = 1 \Leftrightarrow 5^1 = x^2 \Leftrightarrow$ $\qquad x \neq 0$
$\Leftrightarrow x = \pm\sqrt{5}$ ✓

6) $4 \cdot 7^x = 3^{2x} \Rightarrow \log(4 \cdot 7^x) = \log(3^{2x}) \Leftrightarrow$
$\Leftrightarrow \log 4 + \log 7^x = \log 3^{2x} \Leftrightarrow \log 4 + x\log 7 = 2x\log 3$
$\Leftrightarrow \log 4 = x(2\log 3 - \log 7) \Leftrightarrow x = \frac{\log 4}{2\log 3 - \log 7} \Rightarrow$

$x \approx 5.52$

7) $x = \log_9 7 = \frac{\log 7}{\log 9} \approx 0.88$

8) $x = \log_2 (10) = \frac{\log 10}{\log 2} = \frac{1}{\log 2} \approx 3.32$

9) $x = \log_a (b) = \frac{\log b}{\log a}$

EXERCICE 4

1) $\log(x) = \log(3) + \log(4) \Leftrightarrow$ $\qquad x>0$
$\Leftrightarrow \log(x) = \log 12 \Leftrightarrow x = 12$ ✓

2) $\log(x) = \log(7) - \log 6 + \log 48 \Rightarrow$ $\qquad x>0$
$\Rightarrow \log(x) = \log \frac{7 \cdot 48}{6} = \log 56 \Leftrightarrow x = 56$ ✓

3) $\log(x) = 2 - 3\log 2 + \log 4 - 2\log 5 \Leftrightarrow$ $\qquad x>0$
$\Leftrightarrow \log(x) = 2 - \log 2^3 + \log 4 - \log 5^2 \Leftrightarrow$
$\Leftrightarrow \log(x) = \log 10^2 - \log 8 + \log 4 - \log 25 \Leftrightarrow$
$\Leftrightarrow \log(x) = \log \frac{100 \cdot 4}{8 \cdot 25} = \log 2 \Leftrightarrow x = 2$

EXERCICE 5

$P_0 = 10^5 \qquad 20 \text{ min} \to \frac{1}{3} h \qquad P(t) = 10^5 \cdot e^{at}$

$P(\frac{1}{3}) = 2P(0) \Leftrightarrow \cancel{10^5} \cdot e^{\frac{a}{3}} = 2 \cdot \cancel{10^5} \Leftrightarrow e^{a/3} = 2 \Leftrightarrow \frac{a}{3} = \ln 2 \Leftrightarrow$

$a = 3\ln 2 \approx 2.07$

$P(t) = 10^5 \cdot e^{2.07t}$

1) $t = 6 \quad P(6) = 10^5 \cdot e^{2.07 \cdot 6} \simeq 2.48 \cdot 10^{10}$

2) $t = 148 \quad P(148) = 10^5 \, e^{2.07 \cdot 148}$

3) $P(t) = 10^9 \Rightarrow 10^5 e^{2.07 \cdot t} = 10^9 \Leftrightarrow e^{2.07 \cdot t} = 10^4 \Leftrightarrow$

$2.07 \cdot t = \ln 10^4 \Leftrightarrow t \approx 4,45 \text{ h}$

EXERCICE 7

a) $f_1(x) = e^x$

- $D_{f_1} = \mathbb{R}$
-

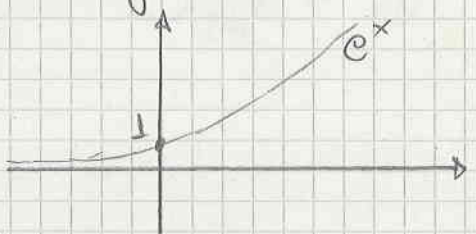
x	$-\infty$		$+\infty$
f_1		$+$	

- $f_1'(x) = e^x = 0$ impossible
  $f_1(x) > 0 \quad \forall x \in \mathbb{R} \Rightarrow f_1 \nearrow$

x	$-\infty$		$+\infty$
f_1'		$+$	
f_1		$\nearrow$	

- $f_1''(x) = e^x > 0$

x	$-\infty$		$+\infty$
f_1''		$+$	
f		$\cup$	



b) $f_2(x) = e^x - 1 \qquad D_f = \mathbb{R}$

- $f_2(x) = 0 \Leftrightarrow e^x = 1 \Rightarrow x = 0$.

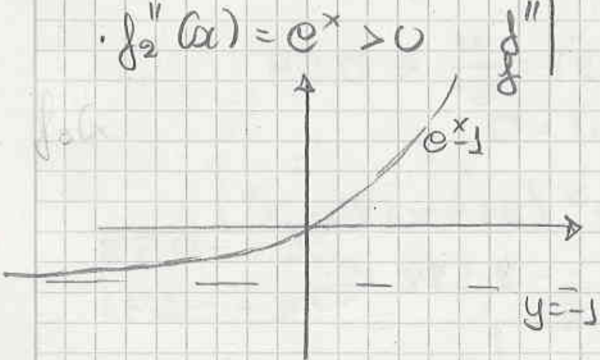
x	$-\infty$		0		$+\infty$
f_2		$-$	0	$+$	

- $f_2'(x) = e^x > 0$

x	$-\infty$		$+\infty$
f_2'		$+$	
f_2		$\nearrow$	

- $f_2''(x) = e^x > 0$

	$-\infty$		$+\infty$
f''		$+$	
f		$\cup$	



c) $f_3(x) = e^x - x \qquad D_{f_3} = \mathbb{R}$

· $f_3(x) = 0 \Leftrightarrow e^x = x$ impossible

x	$-\infty$		$+\infty$
f_3		$+$	

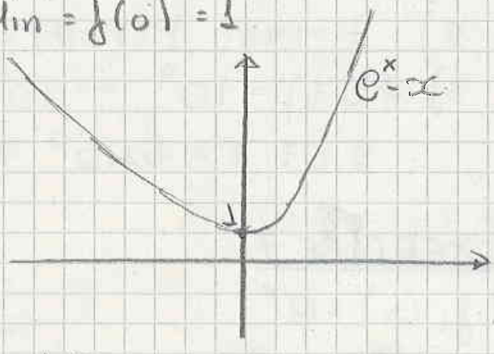
· $f_3'(x) = e^x - 1 = 0 \Leftrightarrow x = 0$

x	$-\infty$		0		$+\infty$
f_3'		$-$	0	$+$	
f_3		$\searrow$		$\nearrow$	

$Min = f(0) = 1$

· $f_3''(x) = e^x > 0$

x	$-\infty$		$+\infty$
f_3''		$+$	
f_3		$\cup$	



d) $f_4(x) = xe^x \qquad D_f = \mathbb{R}$

x	$-\infty$		0		$+\infty$
f_4		$-$	0	$+$	

· $f_4(x) = 0 \Leftrightarrow x = 0$

· $f_4'(x) = e^x + xe^x = e^x(x+1) = 0 \Leftrightarrow x = -1$

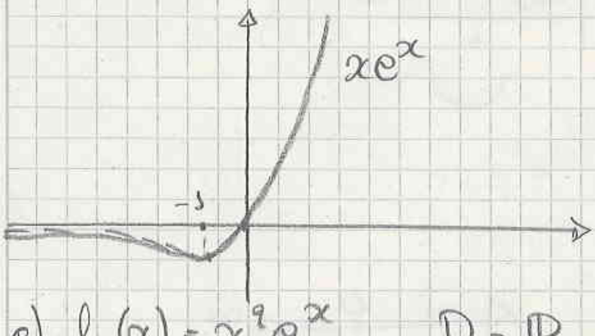
x	$-\infty$		-1		$+\infty$
f_4'		$-$	0	$+$	
f_4		$\searrow$		$\nearrow$	

$Min = f(-1) = -\frac{1}{e} = -0{,}37$

· $f_4''(x) = e^x + e^x + xe^x = e^x(2+x) = 0 \Leftrightarrow$

$\Leftrightarrow x = -2$

x	$-\infty$		-2		$+\infty$
f_4''		$-$	0	$+$	
f_4		$\cap$	PI	$\cup$	



e) $f_5(x) = x^2 e^x \qquad D_f = \mathbb{R}$

x	$-\infty$		0		$+\infty$
f_5		$+$	0	$+$	

· $f_5(x) = 0 \Leftrightarrow x = 0$

$f_5'(x) = 2xe^x + x^2e^x = e^x x(2+x) = 0 \Leftrightarrow x = 0 \quad x = -2$

	$-\infty$		-2		0		$+\infty$
f_5'		$+$	0	$-$	0	$+$	
f_5		$\nearrow$	Max	$\searrow$	Min	$\nearrow$	

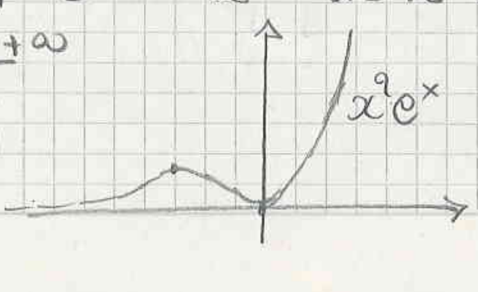
$Max = f(-2) = \frac{4}{e^2} \approx 0{,}54$

$Min = f(0) = 0$

$f_5''(x) = 2e^x + 2xe^x + 2xe^x + x^2e^x = e^x(2 + 2x + 2x + x^2) =$

$= e^x(x^2 + 4x + 2) = 0 \Leftrightarrow x = -2 \pm \sqrt{2} \left\langle \begin{array}{l} x_1 = -0.58 \\ x_2 = -3{,}41 \end{array} \right.$

	$-\infty$		$-3{,}41$		-0.58		$+\infty$
f_5''		$+$	0	$-$	0	$+$	
f_5		$\cup$	PI	$\cap$	PI	$\cup$	



EXERCICE 8

$f(x) = e^{-x^2} \Rightarrow f'(x) = -2x \cdot e^{-x^2}$

EXERCICE 9

$f(x) = e^{-kx^2}$ $\qquad x = \sqrt{3}$

$f'(x) = -2kx \cdot e^{-kx^2}$

$f''(x) = -2k\, e^{-kx^2} - 2kx \cdot (-2kx)\, e^{-kx^2} =$

$= -2k\, e^{-kx^2}(1 - 2kx^2) = 0 \Leftrightarrow \begin{cases} k = 0 \\ 1 = 2kx^2 \end{cases}$

$x = \sqrt{3}$ $\quad 1 = 2k \cdot 3 \Rightarrow k = \frac{1}{6}$ $\qquad k_1 = 0 \quad k_2 = \frac{1}{6}$

EXERCICE 10

1) $f(x) = \ln(x)$ $\qquad D_f =]0, +\infty[$

$f(x) = 0 \Leftrightarrow \ln(x) = 0 \Leftrightarrow x = 1$

x	$-\infty$		1		$+\infty$
$f(x)$		$-$	0	$+$	

2) $f(x) = \ln(x^2)$ $\qquad D_f = \mathbb{R} \setminus \{0\}$

$f(x) = 0 \Leftrightarrow \ln(x^2) = 0 \Leftrightarrow x^2 = 1 \Leftrightarrow x = \pm 1$

x	$-\infty$		-1		1		$+\infty$
f		$+$	0	$-$	0	$+$	

3) $f(x) = \ln(x^2 - 1)$ $\qquad x^2 - 1 > 0$ $\qquad D_f =]-\infty; -1[\cup]1, +\infty[$

$f(x) = 0 \Leftrightarrow \ln(x^2 - 1) = 0 \Leftrightarrow x^2 - 1 = 1 \Leftrightarrow x^2 = 2 \Leftrightarrow x = \pm\sqrt{2}$

	$-\infty$		$-\sqrt{2}$		-1		1		$\sqrt{2}$		$+\infty$
f		$+$	0	$-$		//////		$-$	0	$+$	

4) $f(x) = \ln\left(\frac{1}{6}x^2 + \frac{1}{6}x - 1\right)$ $\qquad \frac{1}{6}x^2 + \frac{1}{6}x - 1 > 0 \Leftrightarrow x^2 + x - 6 > 0$

$(x+3)(x-2) > 0$

$D_f =]-\infty; -3[\cup]2, +\infty[$

$f(x) = 0 \Leftrightarrow \ln\left(\frac{1}{6}x^2 + \frac{1}{6}x - 1\right) = 0$

$\Leftrightarrow \frac{1}{6}x^2 + \frac{1}{6}x - 1 = 1 \Leftrightarrow$

$\Leftrightarrow x^2 + x - 12 = 0 \Leftrightarrow x = \frac{-1 \pm 7}{2} < \begin{matrix} x_1 = -4 \\ x_2 = 3 \end{matrix}$

	$-\infty$		-4		-3		2		3		$+\infty$
f		$+$	0	$-$		//////		$-$	0	$+$	

EXERCICE 11

$f(x) = \ln\left(\dfrac{x^2-1}{x}\right)$ $\qquad \dfrac{x^2-1}{x} > 0 \Leftrightarrow (x-1)(x+1)x > 0$

	$-\infty$		-1		0		1		$+\infty$
x^2-1		+	0	−		−	0	+	
x		−		−	0	+		+	
$\dfrac{x^2-1}{x}$		−	0	+	‖	−	0	+	

$D_f =]-1, 0[\cup]1, +\infty[$

$f'(x) = \dfrac{1}{\frac{x^2-1}{x}} \cdot \dfrac{2x \cdot x - (x^2-1)}{x^2} =$

$= \dfrac{\cancel{x}}{x^2-1} \cdot \dfrac{2x^2-x^2+1}{x^2} =$

$= \dfrac{x^2+1}{x(x^2-1)} = 0$ impossible

x	$-\infty$		-1		0		1		$+\infty$
f'			‖	+	‖		‖	+	
f				↗				↗	

EXERCICE 12

1) $f(x) = x + \ln(x)$ $\qquad D_f =]0, +\infty[$

$\cdot\ f(x) = 0 \Leftrightarrow \ln x = -x \Leftrightarrow x = 0{,}565$

x	0		$0{,}565$		$+\infty$
f	‖	−	0	+	

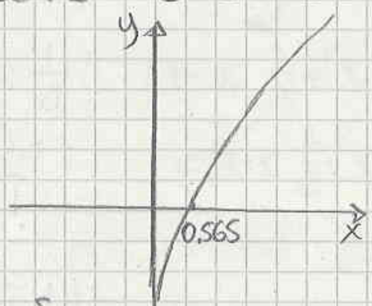
$\cdot\ \lim\limits_{x \to +\infty} \dfrac{f(x)}{x} = 1 + \lim\limits_{x \to +\infty} \dfrac{\ln x}{x} = 1 + 0 = 1$

$\lim\limits_{x \to +\infty} (x + \ln x - x) = \lim\limits_{x \to +\infty} \ln x = +\infty$ $\Big\}$ pas AO / AH

$\cdot\ \lim\limits_{x \to 0} f(x) = -\infty$ $\qquad$ AV : $x = 0$

$\cdot\ f'(x) = 1 + \dfrac{1}{x} = 0 \Leftrightarrow x + 1 = 0 \Leftrightarrow x = -1$ impossible

	0		$+\infty$
f'	‖	+	
f		↗	



2) $y = x - \ln x$ $\qquad D_f =]0, +\infty[$

$\cdot\ f(x) = 0 \Leftrightarrow \ln x = x$ impossible

	0		$+\infty$
f	‖	+	

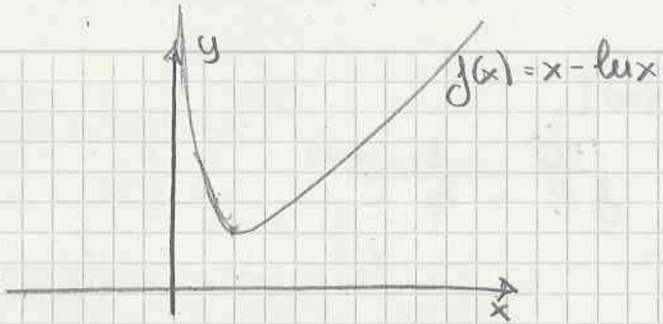
$\cdot\ \lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} x\left(1 - \dfrac{\ln x}{x}\right) = +\infty$ $\qquad$ Pas (AH)

$\cdot\ \lim\limits_{x \to 0} f(x) = +\infty \Rightarrow$ AV $x = 0$

$\cdot\ f'(x) = 1 - \dfrac{1}{x} = 0 \Leftrightarrow x - 1 = 0 \Leftrightarrow x = 1$

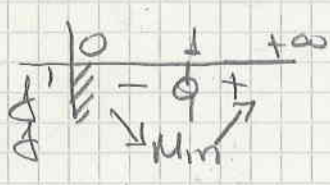
	0		1		$+\infty$
f'	‖	−	0	+	
f	‖	↘	Min	↗	

Min $= f(1) = 1$



5) $f(x) = (\ln x)^2$ $\quad D_f =]0, +\infty[$

- $f(x) = 0 \Leftrightarrow \ln x = 0 \Leftrightarrow x = 1$
- $\lim_{x \to +\infty} f(x) = +\infty$ $\quad \emptyset$ AH
- $\lim_{x \to 0} f(x) = +\infty$ $\quad$ AV : $x = 0$
- $f'(x) = \frac{2 \ln x}{x} = 0 \Leftrightarrow \ln x = 0 \Leftrightarrow x = 1$

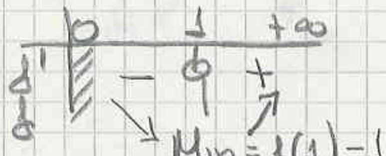


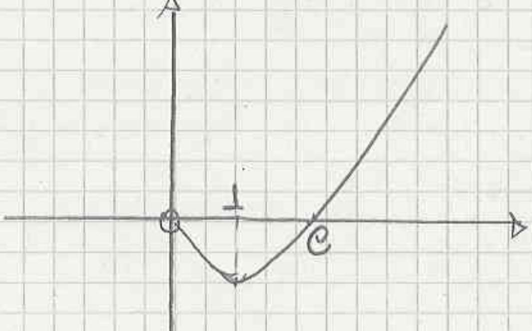
Min $= f(1) = 0$

4) $f(x) = x \ln(x) - x$ $\quad D_f =]0, +\infty[$

- $f(x) = 0 \Leftrightarrow x(\ln x - 1) = 0 \Leftrightarrow \underset{\text{imp}}{x = 0}$ $\quad \ln x = 1 \Leftrightarrow x = e$

- $\lim_{x \to +\infty} f(x) = +\infty$ $\quad$ Pas AH
- $\lim_{x \to 0} f(x) = 0$ $\quad$ Trou $(0,0)$
- $f'(x) = \ln x + \frac{x}{x} - 1 = \ln x = 0 \Leftrightarrow x = 1$





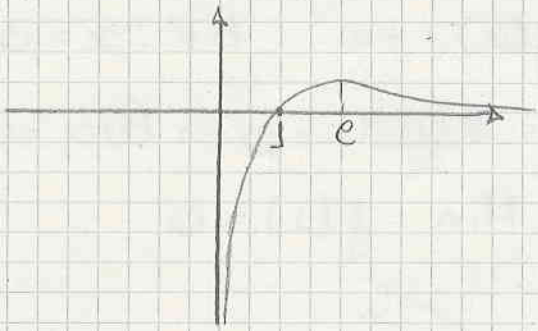
3) $f(x) = \frac{\ln x}{x}$ $\quad D_f =]0, +\infty[$

x	0		1		$+\infty$
f	‖	$-$	0	$+$	

- $f(x) = 0 \Leftrightarrow \ln x = 0 \Leftrightarrow x = 1$
- $\lim_{x \to +\infty} f(x) = 0$ $\quad$ AH: $y = 0$
- $\lim_{x \to 0} f(x) = -\infty$ $\quad$ AV $x = 0$
- $f'(x) = \frac{\frac{1}{x} \cdot x - \ln x}{x^2} = \frac{1 - \ln x}{x^2} = 0 \Leftrightarrow \ln x = 1 \Leftrightarrow x = e$

x	0		e		$+\infty$
f'	‖	$+$	0	$-$	
f		$\nearrow$	Max	$\searrow$	

$\text{Max} = f(e) = \frac{1}{e} \simeq 0.37$



6) $f(x) = x \ln(x)$ $\quad D_f =]0, +\infty[$

x	0		1		$+\infty$
f	‖	$-$	0	$+$	

- $f(x) = 0 \Leftrightarrow \ln x = 0 \Leftrightarrow x = 1$
  $x = 0$ impossible
- $\lim_{x \to +\infty} f(x) = +\infty$ $\quad \emptyset$ AH.
- $\lim_{x \to 0} f(x) = 0$ $\quad$ Trou $(0,0)$
- $f'(x) = \ln x + \frac{x}{x} = 1 + \ln x = 0 \Leftrightarrow \ln x = -1 \Leftrightarrow x = \frac{1}{e} \simeq 0.37$

x	0		$1/e$		$+\infty$
f'	‖	$-$	0	$+$	
f		$\searrow$	Min	$\nearrow$	

$\text{Min} = f\left(\frac{1}{e}\right) = \frac{1}{e}(-1) = -\frac{1}{e} = -0.37$