SERIE 4:Reccurence Solutions

EXERCICE 1

a) 1. $S_1 = a_1 = 1 \quad S_2 = 1+2 = 3 \quad S_3 = 1+2+3 = 6 \quad S_4 = 6+4 = 10$

$$\cancel{a_1} = a_1$$
$$\cancel{a_2} = \cancel{a_1} + 1$$
$$\cancel{a_3} = \cancel{a_2} + 1$$
$$\cancel{a_4} = \cancel{a_3} + 1$$
$$\vdots$$
$$+ \quad a_n = \cancel{a_{n-1}} + 1$$

$$* \boxed{a_n = a_1 + (n-1)\cdot 1} \overset{a_1=1}{\iff} a_n = 1 + n - 1 = \underline{n = a_n}$$

$$S_n = a_1 + (a_1+1) + (a_1+2) + \dots + (a_1 + (n-1))$$
$$+ \quad S_n = a_n + (a_n - 1) + (a_n - 2) + \dots + (a_n - (n-1))$$

$$2S_n = (a_1 + a_n) + (a_1 + a_n) + \dots + (a_1 + a_n) \Rightarrow$$

$$\Rightarrow S_n = \frac{n(a_1 + a_n)}{2} \overset{*}{\iff} S_n = \frac{n(a_1 + a_1 + (n-1))}{2} \Rightarrow$$

$$S_n = \frac{n(2a_1 + (n-1))}{2} \overset{a_1=1}{\Longrightarrow} S_n = \frac{n(n+1)}{2}$$

$$\lim_{n \to +\infty} S_n = +\infty$$

b) $a_1 = 1 \quad a_2 = 3 \quad a_3 = 5. \qquad a_n = a_1 + (n-1)\cdot 2$

$$S_n = \frac{n(2a_1 + (n-1)\cdot 2)}{2} \qquad \lim_{n \to +\infty} S_n = +\infty$$

En général si $a_n = a_{n-1} + w$

$$S_n = \frac{n(2a_1 + (n-1)w)}{2}$$

$n=1 \quad S_1 = \frac{1(2a_1)}{2} = a_1$ ✓

$n=K \quad S_K = \frac{K(2a_1 + (K-1)w)}{2} \underset{Hyp}{=} a_1 + a_2 + \dots + a_K$

$n=K+1 \quad S_{K+1} = \frac{(K+1)(2a_1 + Kw)}{2}$ A montrer

$$S_{K+1} = \underbrace{a_1 + a_2 + \dots + a_K}_{S_K} + a_{K+1} = S_K + a_{K+1} =$$

$$= \frac{k(2a_1+(k-1)w)}{2} + a_k + w =$$

$$= \frac{k(2a_1+(k-1)w) + 2(a_1+(k-1)w) + 2w}{2} =$$

$$= \frac{2ka_1 + k(k-1)w + 2a_1 + 2(k-1)w + 2w}{2}$$

$$= \frac{2ka_1 + k^2w - kw + 2a_1 + 2kw - \cancel{2w} + \cancel{2w}}{2} =$$

$$= \frac{2ka_1 + k^2w + kw + 2a_1}{2} = \frac{k(2a_1+kw) + (2a_1+kw)}{2} =$$

$$= \frac{(2a_1+kw)(k+1)}{2}$$

c) $a_1 = 1 \quad a_2 = 4 \quad a_3 = 9 \quad a_4 = 16 \quad a_n = n^2$

$S_1 = 1 \Rightarrow 6S_1 = 6 = 1 \cdot 2 \cdot 3$

$S_2 = a_1 + a_2 = 5 \Rightarrow 6S_2 = 30 = 2 \cdot 3 \cdot 5$

$S_3 = a_1 + a_2 + a_3 = 14 \Rightarrow 6S_3 = 84 = 3 \cdot 4 \cdot 7$

$S_4 = a_1 + a_2 + a_3 + a_4 = 30 \Rightarrow 6 \cdot S_4 = 180 = 4 \cdot 5 \cdot 9$

$$6S_n = n(n+1)(2n+1) \Rightarrow S_n = \frac{n(n+1)(2n+1)}{6}$$

$n = 1 \quad S_1 = 1$ ✓

$n = k \quad S_k = \frac{k(k+1)(2k+1)}{6}$ Hyp $= a_1 + a_2 + \dots + a_k$

$n = k+1 \quad S_{k+1} = \frac{(k+1)(k+2)(2(k+1)+1)}{6}$ à montrer

$$S_{k+1} = \underbrace{a_1 + \dots + a_k}_{S_k} + a_{k+1} = S_k + a_{k+1} =$$

$$\overset{\text{Hyp}}{=} \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

$$= \frac{(k+1)\left(k(2k+1)+6(k+1)\right)}{6} = \frac{(k+1)\left(2k^2+k+6k+6\right)}{6} =$$

$$= \frac{(k+1)\left(2k^2+7k+6\right)}{6} = \frac{(k+1)\,2(k+2)(k+\frac{3}{2})}{6} \overset{*}{=}$$

$$\Delta = 49-48=1 \qquad k_{1,2} = \frac{-7\pm 1}{4} = \begin{cases} -2 \\ -\frac{6}{4} = -\frac{3}{2} \end{cases}$$

$$* = \frac{(k+1)(k+2)(2k+3)}{6} \quad \checkmark$$

d) $a_1 = \frac{1}{1\cdot 3} \qquad a_2 = \frac{1}{3\cdot 5} \qquad a_3 = \frac{1}{5\cdot 7} \qquad a_n = \frac{1}{(2n-1)(2n+1)}$

$$S_1 = a_1 = \frac{1}{1\cdot 3} = \frac{1}{3} \qquad S_2 = \frac{1}{3} + \frac{1}{15} = \frac{2}{5} \qquad S_3 = \frac{2}{5} + \frac{1}{35} = \frac{3}{7}$$

$$S_n = \frac{n}{2n+1}$$

$n=1 \quad S_1 = \frac{1}{3}$ ✓

$n=k \quad S_k = \frac{k}{2k+1}$ Hyp

$n=k+1 \quad S_{k+1} = \frac{k+1}{2(k+1)+1} = \frac{k+1}{2k+3}$ A montrer

$$S_{k+1} = S_k + a_{k+1} = \frac{k}{2k+1} + \frac{1}{(2(k+1)-1)(2(k+1)+1)} =$$

$$= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} = \frac{k(2k+3)+1}{(2k+1)(2k+3)} =$$

$$= \frac{2k^2+3k+1}{(2k+1)(2k+3)} = \frac{2(k+1)(k+\frac{1}{2})}{(2k+1)(2k+3)} = \frac{(k+1)\cancel{(2k+1)}}{\cancel{(2k+1)}(2k+3)} =$$

$$\Delta = 9-8=1 \qquad k_{1,2} = \frac{-3\pm 1}{4} = \begin{cases} -1 \\ -\frac{1}{2} \end{cases} \qquad = \frac{k+1}{2k+3} \ \checkmark$$

$$\lim_{k\to+\infty} \frac{k+1}{2k+3} = \frac{1}{2} \qquad \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}$$

EXERCICE 2

1. $\frac{1}{2^1}+\frac{2}{2^2}+\frac{3}{2^3}+\dots+\frac{n}{2^n}=\frac{2^{n+1}-n-2}{2^n}$

$n=1 \quad \sum_{n=1}^{1}\frac{n}{2^n}=\frac{1}{2}$ ✓

$n=k \quad \sum_{n=1}^{k}\frac{n}{2^n}=\frac{1}{2^1}+\frac{2}{2^2}+\dots+\frac{n}{2^n}=\frac{2^{n+1}-n-2}{2^n}$ Hyp

$n=k+1 \quad \sum_{n=1}^{k+1}\frac{n}{2^n}=\underbrace{\frac{1}{2^1}+\frac{2}{2^2}+\dots+\frac{n}{2^n}}_{Hyp}+\frac{n+1}{2^{n+1}}=\frac{2^{n+2}-n-3}{2^{n+1}}$ À voir

$\sum_{n=1}^{k+1}\frac{n}{2^n}=\frac{2^{n+1}-n-2}{2^n}+\frac{n+1}{2^{n+1}}=\frac{2(2^{n+1}-n-2)+n+1}{2^{n+1}}=$

$=\frac{2^{n+2}-2n-4+n+1}{2^{n+1}}=\frac{2^{n+2}-n-3}{2^{n+1}}$ ✓

2. $\lim_{n\to+\infty}\frac{2^{n+2}-n-3}{2^{n+1}}=\lim_{n\to+\infty}\frac{2\cdot 2^{n+1}-n-3}{2^{n+1}}=2$

Donc $\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\dots=2$

EXERCICE 3

$1+r+r^2+r^3+\quad+r^{n-1}=\sum_{i=0}^{n-1}r^i=\frac{1-r^n}{1-r}$

$n=1 \quad \sum_{i=0}^{0}r^i=1=\frac{1-r}{1-r}=1$ ✓

$n=k \quad \sum_{i=0}^{k-1}r^i=\frac{1-r^k}{1-r}$ Hyp

$n=k+1 \quad \sum_{i=0}^{k}r^i=\frac{1-r^{k+1}}{1-r}$ À voir

$\sum_{i=0}^{k}r^i=\underbrace{1+r+r^2+\quad+r^{k-1}}_{Hyp}+r^k=\frac{1-r^k}{1-r}+r^k=$

$=\frac{1-r^k+r^k(1-r)}{1-r}=\frac{1-\cancel{r^k}+\cancel{r^k}-r^{k+1}}{1-r}=\frac{1-r^{k+1}}{1-r}$ ✓

2. $|r|<1$ $\lim\limits_{n\to+\infty} r^{k+1} = 0$ Donc

$$\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$$

3. a) $r = \frac{1}{3}$ $\sum\limits_{i=0}^{\infty}\left(\frac{1}{3}\right)^i = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$

b) $r = \frac{1}{2^2}$ $\sum\limits_{i=0}^{\infty}\left(\frac{1}{2^2}\right)^i = \frac{1}{1-\frac{1}{2^2}} = \frac{4}{3}$

c) $r = -\frac{1}{2^2}$ $\sum\limits_{i=0}^{\infty}\left(-\frac{1}{2^2}\right)^i = \frac{1}{1+\frac{1}{4}} = \frac{4}{5}$.

d) $\frac{3}{2} + \frac{3}{2^3} + \frac{3}{2^5} + \frac{3}{2^7} + \dots = \frac{3}{2}\left(1 + \frac{1}{2^2} + \frac{1}{2^4} + \quad\right) =$

$= \frac{\cancel{3}}{2}\ \frac{4}{\cancel{3}} = 2$

EXERCICE 4

$$\cos x \cdot \cos(2x) \cdot \cos(4x) \cdots \cos(2^n x) = \frac{\sin(2^{n+1}x)}{2^{n+1}\sin x}$$

$n=0$ $\cos x = \frac{\sin(2x)}{2\cdot\sin x} = \frac{\cancel{2}\cancel{\sin x}\cos x}{\cancel{2}\cancel{\sin x}} = \cos x$ ✓

$n=k$ $\cos x \cdot \cos(2x) \cdots \cos(2^k x) = \frac{\sin(2^{k+1}x)}{2^{k+1}\sin x}$ Hyp

$n=k+1$ $\cos x \cdot \cos(2x) \dots \cos(2^k x) \cdot \cos(2^{k+1}x) = \frac{\sin(2^{k+2}x)}{2^{k+2}\sin x}$

A now

$$\overbrace{\cos x \cdot \cos(2x) \dots \cos(2^k x)}^{Hyp} \cdot \cos(2^{k+1}x) =$$

$$= \frac{\sin(2^{k+1}x)}{2^{k+1}\sin x}\cos(2^{k+1}x) = \frac{1}{2}\,\frac{\sin(2^{k+2}x)}{2^{k+1}\sin x} = \frac{\sin(2^{k+2}x)}{2^{k+2}\sin x}$$ ✓

EXERCICE 5

$$\sum_{k=1}^{n} k\cdot k! = (n+1)!-1$$

$n=1$ $\sum\limits_{k=1}^{1} k\cdot k! = 1\cdot 1 = 2!-1$ ✓

$n=m \quad \sum_{k=1}^{m} k\cdot k! = (m+1)!-1$ <u>Hyp</u>

$n=m+1$ À voir: $\sum_{k=1}^{m+1} k\cdot k! = (m+2)!-1$

$$\sum_{k=1}^{m+1} k\cdot k! = \sum_{k=1}^{m} k\cdot k! + (m+1)\cdot(m+1)! \stackrel{Hyp}{=}$$

$$(m+1)!-1+(m+1)\cdot(m+1)! = (m+1)!\,(1+m+1)-1 =$$

$$=(m+1)!\,(m+2)-1 = (m+2)!-1 \checkmark$$

EXERCICE 6

1) $12 \mid 10^n-4 \quad n\geq 2$

$n=2 \quad 10^2-4 = 96 = 8\cdot 12 \checkmark$

$n=k \quad 10^k-4 = 12\cdot m$ <u>Hyp</u> $\quad m, z \in \mathbb{Z}$

$n=k+1$ À voir $10^{k+1}-4 = 12\cdot z$

$$10^{k+1}-4 = 10^k\cdot 10-4 = (12\cdot m+4)\cdot 10-4 =$$
$$= 120m+40-4 = 120m+36 = 12\,(\underbrace{10m+3}_{z\in\mathbb{Z}})$$

Donc $12 \mid 10^{k+1}-4$

2) $3 \mid 4^n+5$

$n=1 \quad 4+5=9=3\cdot 3 \checkmark$

$n=k \quad 4^k+5 = 3\cdot m$ <u>Hyp</u>

$n=k+1 \quad 4^{k+1}+5 = 3\cdot l$

$$4^{k+1}+5 = 4^k\cdot 4+5 = (3m-5)\,4+5 =$$

$$= 12m-20+5 = 12m-15 = 3\,(\underbrace{4m-5}_{l\in\mathbb{Z}}) \checkmark$$

EXERCICE 7

$$a_n = \frac{1}{(6n-5)(6n+1)}$$

1. $a_1 = \frac{1}{1\cdot 7} \quad a_2 = \frac{1}{7\cdot 13} \quad a_3 = \frac{1}{13\cdot 19}$

2. $S_n = a_1 + a_2 + \ldots + a_n$

$$S_1 = a_1 = \frac{1}{7}$$

$$S_2 = a_1 + a_2 = \frac{1}{7} + \frac{1}{7\cdot 13} = \frac{13+1}{7\cdot 13} = \frac{14}{91} = \frac{2}{13}$$

$$S_3 = a_1 + a_2 + a_3 = \frac{2}{13} + \frac{1}{13\cdot 19} = \frac{2\cdot 19+1}{13\cdot 19} = \frac{39}{13\cdot 19} = \frac{3}{19}$$

$$S_n = \frac{n}{6n+1}$$

3. $n=1 \quad S_1 = \frac{1}{7}$ ✓

$n=k \quad S_k = \frac{k}{6k+1}$ Hyp

$n=k+1 \quad S_{k+1} = \frac{k+1}{6(k+1)+1} = \frac{k+1}{6k+7}$ À voir

$$S_{k+1} = S_k + a_{k+1} = \frac{k}{6k+1} + \frac{1}{(6(k+1)-5)(6(k+1)+1)} =$$

$$= \frac{k}{6k+1} + \frac{1}{(6k+1)(6k+7)} = \frac{k(6k+7)+1}{(6k+1)(6k+7)} =$$

$$= \frac{6k^2+7k+1}{(6k+1)(6k+7)} = \frac{6(k+1)(k+\frac{1}{6})}{(6k+1)(6k+7)} = \frac{(k+1)(\cancel{6k+1})}{(\cancel{6k+1})(6k+7)} = \frac{k+1}{6k+7}$$ ✓

$$\Delta = 49 - 24 = 25 \quad k_{1,2} = \frac{-7 \pm 5}{12} = \begin{cases} -1 \\ -\frac{2}{12} = -\frac{1}{6} \end{cases}$$

4. $S = a_1 + a_2 + a_3 + \ldots = \lim_{k\to\infty} \frac{k+1}{6k+7} = \frac{1}{6}.$

EXERCICE 8

1. $1+7+13+19+\ldots+(6n-5)=n(3n-2)$

$$\sum_{i=1}^{n}(6i-5)=n(3n-2)$$

$n=1$ $\quad \sum_{i=1}^{1}(6i-5)=1=1\cdot 1$ ✓

$n=k$ $\quad \sum_{i=1}^{k}(6i-5)=k(3k-2)$ $\quad$ <u>Hyp</u>

$n=k+1$ $\quad \sum_{i=1}^{k+1}(6i-5)=(k+1)(3(k+1)-2)=(k+1)(3k+1)$ $\quad$ <u>A voir</u>

$$\sum_{i=1}^{k+1}(6i-5)=\sum_{i=1}^{k}(6i-5)+6(k+1)-5=$$

$$=k(3k-2)+6k+6-5=3k^2-2k+6k+1=$$

$$=3k^2+4k+1=3\left(k+\tfrac{1}{3}\right)(k+1)=(3k+1)(k+1) \checkmark$$

$\Delta=16-12=4 \quad k_{1,2}=\dfrac{-4\pm 2}{6}=\begin{cases}-1\\ \dfrac{-2}{6}=-\dfrac{1}{3}\end{cases}$

2. $1+5+5^2+5^3+\ldots+5^{n-1}=\dfrac{5^n-1}{4}$

$$\sum_{i=1}^{n}5^{i-1}=\frac{5^n-1}{4}$$

$n=1$ $\quad \sum_{i=1}^{1}5^{i-1}=5^0=1=\dfrac{5^1-1}{4}=1$ ✓

$n=k$ $\quad \sum_{i=1}^{k}5^{i-1}=\dfrac{5^k-1}{4}$ $\quad$ <u>Hyp</u>

$n=k+1$ $\quad \sum_{i=1}^{k+1}5^{i-1}=\dfrac{5^{k+1}-1}{4}$ $\quad$ A voir

$$\sum_{i=1}^{k+1}5^{i-1}=\sum_{i=1}^{k}5^{i-1}+5^k=\frac{5^k-1}{4}+5^k=\frac{5^k-1+4\cdot 5^k}{4}=$$

$$=\frac{5\cdot 5^k-1}{4}=\frac{5^{k+1}-1}{4}$$

3. $\frac{5}{1\cdot2\cdot3}+\frac{6}{2\cdot3\cdot4}+\dots+\frac{n+4}{n(n+1)(n+2)}=\sum_{i=1}^{n}\frac{i+4}{i(i+1)(i+2)}=\frac{n(3n+7)}{2(n+1)(n+2)}$

$n=1$. $\frac{5}{1\cdot2\cdot3}=\frac{1\cdot(3\cdot1+7)}{2\cdot2\cdot3}=\frac{10}{2\cdot2\cdot3}=\frac{5}{2\cdot3}$ ✓

$n=k$ $\sum_{i=1}^{k}\frac{i+4}{i(i+1)(i+2)}=\frac{k(3k+7)}{2(k+1)(k+2)}$ Hyp

$n=k+1$ $\sum_{i=1}^{k+1}\frac{i+4}{i(i+1)(i+2)}=\frac{(k+1)(3k+10)}{2(k+2)(k+3)}$ A voir

$\sum_{i=1}^{k+1}\frac{i+4}{i(i+1)(i+2)}=\sum_{i=1}^{k}\frac{i+4}{i(i+1)(i+2)}+\frac{k+5}{(k+1)(k+2)(k+3)}\overset{Hyp}{=}$

$=\frac{k(3k+7)}{2(k+1)(k+2)}+\frac{k+5}{(k+1)(k+2)(k+3)}=\frac{(k+3)k(3k+7)+2k+10}{2(k+1)(k+2)(k+3)}=$

$=\frac{3k^3+9k^2+7k^2+21k+2k+10}{2(k+1)(k+2)(k+3)}=\frac{3k^3+16k^2+23k+10}{2(k+1)(k+2)(k+3)}$ (1)

$P(k)=3k^3+16k^2+23k+10$

$P(-1)=0$

3	16	23	10	-1
↓	-3	-13	-10	
3	13	10	0	

$P(k)=(k+1)(3k^2+13k+10)$

$\Delta=49$ $k_{1,2}=\frac{-13\pm7}{6}=$ $-\frac{20}{6}=-\frac{10}{3}$; $-\frac{6}{6}=-1$

$P(k)=(k+1)3\left(k+\frac{10}{3}\right)(k+1)=$
$=(k+1)^2(3k+10)$

$(1)=\frac{(k+1)^{\cancel{2}}(3k+10)}{2\cancel{(k+1)}(k+2)(k+3)}=\frac{(k+1)(3k+10)}{2(k+2)(k+3)}$ ✓

4. $1^2+3^2+5^2+\dots+(2n-1)^2 = \sum_{i=1}^{n} (2i-1)^2 = \frac{n(4n^2-1)}{3}$

$n=1 \quad \sum_{i=1}^{1} (2i-1)^2 = 1 = \frac{1\cdot(4-1)}{3} = 1$ ✓

$n=k \quad \sum_{i=1}^{k} (2i-1)^2 = \frac{k(4k^2-1)}{3}$ Hyp

$n=k+1 \quad \sum_{i=1}^{k+1} (2i-1)^2 = \frac{(k+1)(4(k+1)^2-1)}{3} = \frac{(k+1)(4k^2+8k+4-1)}{3} =$

$= \frac{(k+1)(4k^2+8k+3)}{3}$ À voir

$$\sum_{i=1}^{k+1} (2i-1)^2 = \sum_{i=1}^{k} (2i-1)^2 + (2(k+1)-1)^2 =$$

$$\overset{Hyp}{=} \frac{k(4k^2-1)}{3} + (2k+1)^2 = \frac{k(4k^2-1)+3(4k^2+4k+1)}{3} =$$

$$= \frac{4k^3-k+12k^2+12k+3}{3} = \frac{4k^3+12k^2+11k+3}{3} =$$

$$\begin{array}{cccc|c}
4 & 12 & 11 & 3 & -1 \\
\downarrow & -4 & -8 & -3 & \\
\hline
4 & 8 & 3 & 0 &
\end{array}$$

$$= \frac{(k+1)(4k^2+8k+3)}{3}.$$ ✓