SERIE 3: Primitives - Integrales Solutions

EXERCICE 1

a) $f(x) = a(x-0)^2 + 3 = ax^2 + 3$

$(4;2) \quad 2 = a \cdot 16 + 3 \Leftrightarrow a \cdot 16 = -1 \Leftrightarrow a = -1/16$

$f(x) = -\frac{1}{16}x^2 + 3$

$V = \pi \int_{-4}^{4} (-\frac{1}{16}x^2 + 3)^2 dx = \pi \frac{288}{5}$

b) $x^2 + y^2 = r^2 \Leftrightarrow y^2 = r^2 - x^2 \Leftrightarrow y = \pm\sqrt{r^2 - x^2} \qquad y \geq 0$

$A = \pi \int_{-r}^{r} (r^2 - x^2) dx = \pi \cdot \frac{16r^5}{15}$

c) $y = ax + b \quad a = \frac{r}{h} \quad b = 0$

$y = \frac{r}{h}x$

$V = \pi \int_0^r (\frac{r}{h}x)^2 dx = \frac{\pi r^2}{h^2} \int_0^r x^2 dx = \frac{\pi r^2}{h^2} \frac{r^3}{3} = \frac{\pi r^5}{3h^2}$

e) $V = \pi \int_1^4 \frac{4}{x^2} dx = 3\pi$

f) $V = \pi \int_0^{2\pi} (2 + \sin x)^2 dx = \pi \int_0^{2\pi} (4 + 4\sin x + \sin^2 x) dx =$

$= 4\pi \int_0^{2\pi} dx + 4\pi \int_0^{2\pi} \sin x \, dx + \pi \int_0^{2\pi} \sin^2 x \, dx = 9\pi^2$

$* \int \sin^2 x \, dx = \int \underset{u'}{\sin x} \cdot \underset{v}{\sin x} = -\cos x \cdot \sin x - \int ((-\cos x)\cos x) dx =$

$= -\cos x \sin x + \int (1 - \sin^2 x) dx = -\cos x \sin x + \int dx - \int \sin^2 x \, dx \Leftrightarrow$

$$\Leftrightarrow 2\int \sin^2 x\,dx = -\cos x \sin x + x \Leftrightarrow$$

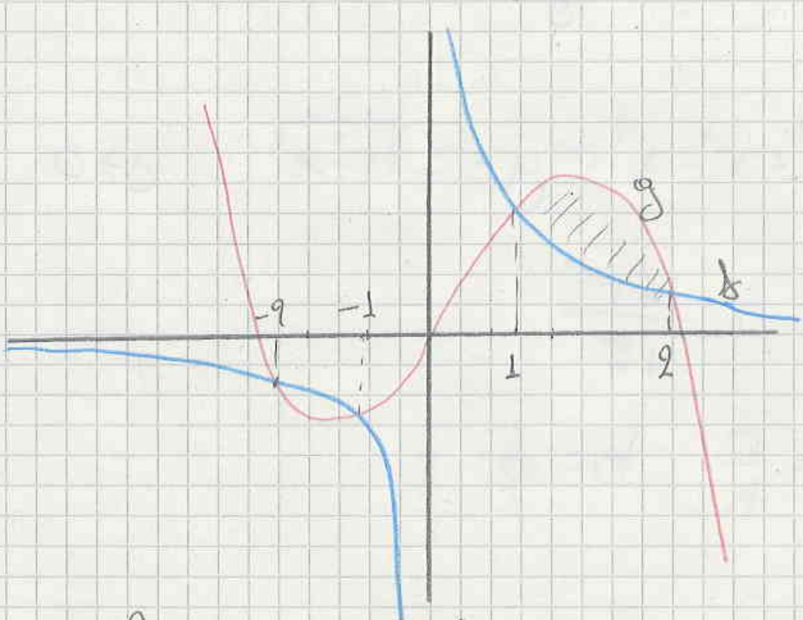
$$\int \sin^2 x\,dx = \frac{1}{2}\left(-\cos x \sin x + x\right)$$

EXERCICE 2

$$f(x) = g(x) \Leftrightarrow \frac{4}{x} = 5x - x^3 \Leftrightarrow 4 = 5x^2 - x^4 \Leftrightarrow x^4 - 5x^2 + 4 = 0$$

$$t^2 - 5t + 4 = 0$$

$$t_{1,2} = \begin{cases} 4 = x^2 \Leftrightarrow x_1 = 2 \quad x_2 = -2 \\ 1 = x^2 \Leftrightarrow x_3 = 1 \quad x_4 = -1 \end{cases}$$



$$V = \pi \int_1^2 (g - f)^2\,dx = \pi \int_1^2 \left(5x - x^3 - \frac{4}{x}\right)^2 dx = \frac{\pi \cdot 8}{7}$$

EXERCICE 3

i) $f(x) = x \cdot a^x$

$$\int x a^x\,dx = \frac{x a^x}{\ln a} - \int \frac{a^x}{\ln a}\,dx = \frac{x a^x}{\ln a} - \frac{a^x}{\ln^2 a} + c$$

$u = x \qquad u' = 1$

$v' = a^x \qquad v = \frac{a^x}{\ln a}$

ii) $\int \ln x \, dx = \int (x)' \ln x \, dx = x \ln x - \int x \frac{1}{x} dx = x \ln x - \int dx$

$= x \ln x - x + c$

iii) $\int e^x \sin x \, dx = \int (e^x)' \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx =$

$= e^x \sin x - \int (e^x)' \cos x \, dx =$

$= e^x \sin x - \left[e^x \cos x + \int e^x \sin x \, dx \right] \Leftrightarrow$

$\Leftrightarrow \int e^x \sin x \, dx = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx \Leftrightarrow$

$\Leftrightarrow \int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x)$

iv) $\int x \sqrt{x+1} \, dx = \int x \left(\frac{2}{3} (x+1)^{2/3} \right)' dx =$

$u = x \qquad u' = 1$

$v' = \sqrt{x+1} \qquad v = \frac{2}{3} (x+1)^{2/3}$

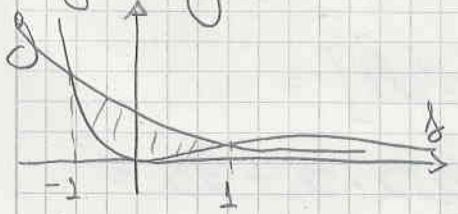
$= \frac{2}{3} x (x+1)^{2/3} - \frac{2}{3} \int (x+1)^{2/3} dx = \frac{2}{3} x (x+1)^{2/3} - \frac{2}{\cancel{3}} \cdot \frac{\cancel{3}}{5} (x+1)^{5/3} =$

$= \frac{2}{3} x (x+1)^{2/3} - \frac{2}{5} (x+1)^{5/3} + c$

EXERCICE 4

i) $f(x) = x^2 e^{-x} \qquad g(x) = e^{-x}$

$f(x) = g(x) \Leftrightarrow x^2 e^{-x} = e^{-x} \Leftrightarrow e^{-x} (x^2 - 1) = 0 \Leftrightarrow x = \pm 1$



$A = \int_{-1}^{1} (g - f) = \int_{-1}^{1} e^{-x} (1 - x^2) dx =$

$u = 1 - x^2 \qquad u' = -2x$

$v' = e^{-x} \qquad v = -e^{-x}$

$= -e^{-x} (1 - x^2) \Big|_{-1}^{1} - \int_{-1}^{1} (-2x)(-e^{-x}) dx = -e^{-x} (1 - x^2) \Big|_{-1}^{1} - 2 \int_{-1}^{1} x e^{-x} dx$

Donc $A = [x\ln x - x]_1^e - [x\ln^2 x + 2x\ln x + 2x]_1^e = 3 - e$

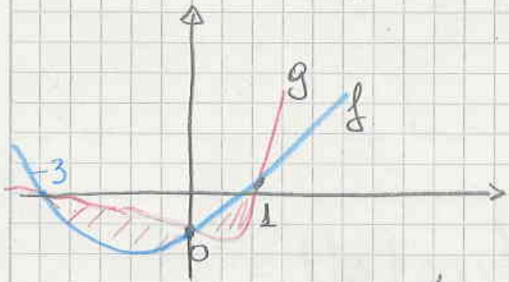
$=$

iii) $f(x) = x^2 + 2x - 3 \qquad g(x) = (x^2+2x-3)e^x$

$f(x) = g(x) \Leftrightarrow (x^2+2x-3)(e^x - 1) = 0 \Leftrightarrow$

$x^2 + 2x - 3 = 0 \quad < \begin{matrix} x_1 = -3 \\ x_2 = 1 \end{matrix}$

$e^x = 1 \Leftrightarrow x_3 = 0$



$A = \int_{-3}^{0}(g-f) + \int_0^1 f - g = \int_{-3}^{0}(x^2+2x-3)e^x - (x^2+2x-3)\,dx +$

$+ \int_0^1 (x^2+2x-3) - (x^2+2x-3)e^x\,dx = 6 - \frac{6}{e^3} + 2e - \frac{14}{3}$

$\int (x^2+2x-3)(e^x-1)\,dx = \int (x^2+2x-3)(e^x - x)'\,dx =$

$= (x^2+2x-3)(e^x-x) - \int (2x+2)(e^x-x)\,dx =$

$= (x^2+2x-3)(e^x-x) - 2\int (x+1)(e^x - \frac{1}{2}x^2)'\,dx =$

$= (x^2+2x-3)(e^x-x) - 2\left[(x+1)(e^x - \frac{x^2}{2}) - \int (e^x - \frac{1}{2}x^2)\,dx\right] =$

$= (x^2+2x-3)(e^x-x) - 2(x+1)(e^x - \frac{x^2}{2}) + 2(e^x - \frac{1}{6}x^3)$

Exercice 5

i) $f(x) = x(1-x^2)^4$

$t = 1 - x^2 \Rightarrow dt = -2x\,dx$

$\int f(x)dx = \int \cancel{x} \cdot t^4 \cdot \left(-\frac{1}{2\cancel{x}}\right) dt = -\frac{1}{2}\int t^4 dt = -\frac{1}{2}\frac{1}{5}t^5 =$

$= -\frac{1}{10}(1-x^2)^5 + c$

ii) $f(x) = x\cos(\pi x^2)$

$t = \pi x^2 \Rightarrow dt = 2\pi x\,dx$

$\int f(x)dx = \int \cancel{x}\cos t \cdot \frac{dt}{2\pi\cancel{x}} = \frac{1}{2\pi}\int \cos t\,dt =$

$= \frac{1}{2\pi}\sin t = \frac{1}{2\pi}\sin(\pi x^2) + c$

iii) $f(x) = x^3\sqrt{1+x^4}$

$\sqrt{1+x^4} = t \Rightarrow 1 + x^4 = t^2 \Leftrightarrow 4x^3dx = 2t\,dt$

$\Leftrightarrow x^3dx = \frac{1}{2}t\,dt$

$\int f(x)dx = \int \frac{1}{2}t \cdot t\,dt = \frac{1}{2}\int t^2dt = \frac{1}{6}t^3 = \frac{1}{6}(1+x^4)^{3/2} + c$

iv) $f(x) = \frac{\ln x}{x}$

$\ln x = t \Rightarrow \frac{1}{x}dx = dt$

$\int \frac{\ln x}{x}dx = \int t\,dt = \frac{1}{2}t^2 = \frac{1}{2}\ln^2 x + c$

v) $f(x) = \frac{2x+1}{x^2+x+4} = \frac{(x^2+x+4)'}{(x^2+x+4)}$

$\int f(x)dx = \ln(x^2+x+4) + c$

vi) $f(x) = \frac{1}{\sqrt{1-9x^2}}$

$-x = \frac{1}{3}\sin t \Rightarrow dx = \frac{1}{3}\cos t\, dt$

$3x = \sin t \Rightarrow t = \mathrm{Arcsin}(3x)$

$\int f(x)dx = \int \frac{1}{\sqrt{1-9\cdot\frac{1}{9}\sin^2 t}} \frac{1}{3}\cos t\, dt =$

$= \frac{1}{3}\int \frac{\cos t\, dt}{\sqrt{1-\sin^2 t}} = \frac{1}{3}\int \frac{\cos t\, dt}{\cos t} = \frac{1}{3}\int dt = \frac{1}{3}t =$

$= \frac{1}{3}\mathrm{Arcsin}(3x) + c$

EXERCICE 6

$f(x) = \frac{-x^2+4}{(x+1)^2} = A + \frac{B}{x+1} + \frac{C}{(x+1)^2} \Leftrightarrow$

$\Leftrightarrow -x^2+4 = A(x+1)^2 + B(x+1) + C \Leftrightarrow$

$\Leftrightarrow -x^2+4 = Ax^2 + 2Ax + A + Bx + B + C =$

$= Ax^2 + (2A+B)x + A+B+C$

$A = -1$

$2A + B = 0 \Leftrightarrow B = 2$

$A+B+C = 4 \Rightarrow -1+2+C = 4 \Leftrightarrow C = 4-1 \Leftrightarrow C = 3$

$\int f(x)dx = \int(-1)dx + 2\int\frac{dx}{x+1} + 3\int\frac{dx}{(x+1)^2} =$

$= -x + 2\ln(x+1) + 3\left(\frac{1}{-1}\right)(x+1)^{-1} =$

$= -x + 2\ln(x+1) - \frac{3}{x+1} + c$