LDDR- Niveau II: Taylor

EXERCICE 1

$f(x) = x^3$

a) $a = 1$

$f(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2$

$f'(x) = 3x^2 \quad f'(1) = 3 \qquad f(1) = 1$

$f''(x) = 6x \quad f''(1) = 6$

$f(x) = 1 + \frac{3}{1!}(x-1) + \frac{6}{2!}(x-1)^2$

$f(x) = 1 + 3(x-1) + 3(x-1)^2$

$R_2(x) = \int_1^x \frac{f^{(3)}(t)}{3!}(x-t)^2 dt$

$f^{(3)}(x) = 6$ donc $R_2(x) = \int_1^x \frac{6}{6}(x-t)^2 dt =$

$= \int_1^x (x-t)^2 dt = -\frac{1}{3}(x-t)^3\Big]_1^x = -\frac{1}{3}(x-x)^3 + \frac{1}{3}(x-1)^3 = \frac{1}{3}(x-1)^3$

b) $f(x) = \left(x+\frac{1}{2}\right)^3 - \frac{3}{2}\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}\left(x+\frac{1}{2}\right) - \frac{1}{8}$

c) $f(x) = (2x+1)^3$

$f(1) = 3^3 = 27$

$f'(x) = 3(2x+1)^2 \cdot 2 = 6(2x+1)^2 \qquad f'(1) = 54$

$f''(x) = 12 \cdot 2(2x+1) = 24(2x+1) \qquad f''(1) = 72$

$f(x) = 27 + \frac{54}{1!}(x-1) + \frac{72}{2!}(x-1)^2 \Rightarrow$

$f(x) = 27 + 54(x-1) + 36(x-1)^2$

$f^{(3)}(x) = 24 \cdot 2 = 48$

$R_2(x) = \int_1^x \frac{f^{(3)}(t)}{3!}(x-t)^2 dt = \int_1^x \frac{48}{6}(x-t)^2 dt = -8\,\frac{1}{3}(x-t)^3\Big]_1^x =$

$= -\frac{8}{3}(x-x)^3 + \frac{8}{3}(x-1)^3 \Rightarrow R_2(x) = \frac{8}{3}(x-1)^3$

Ordre 3 , $a = -\frac{1}{2}$

$$f(-\tfrac{1}{2}) = f'(-\tfrac{1}{2}) = f''(-\tfrac{1}{2}) = 0$$

Donc $\boxed{f(x) = 8\left(x+\frac{1}{2}\right)^3}$

EXERCICE 2

a) $f(x) = \cos(x)$

$f(0) = 1$ $\quad k=0$

$f'(x) = -\sin x \quad f'(0) = 0$

$f''(x) = -\cos x \quad f''(0) = -1 \quad k=1$

$f'''(x) = \sin x \quad f'''(0) = 0$

Donc $f^{(2k)}(x) = (-1)^k$

$$\boxed{\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{27} - \frac{x^6}{720} + \dots \frac{(-1)^k x^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = \cos x}$$

$$R_n(x) = R_{2k}(x) = \frac{f^{(2k+1)}(c)}{(2k+1)!} x^{2k+1} \qquad c \in [0, x].$$

Comme $f^{(2k+1)}(x)$ est un cos ou sin

$$|f^{(2k+1)}(c)| \leq 1$$

Donc $\boxed{|R_{2k}(x)| \leq \left|\frac{x^{2k+1}}{(2k+1)!}\right|} \to 0$

b) $f(x) = e^x$

$f(0) = 1$

$f'(x) = e^x \quad f'(0) = 1 \qquad f^{(n)}(x) = e^x \quad f^{(n)}(0) = 1$

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots = \sum_{k=0}^{\infty} \frac{x^k}{k!}}$$

$R_n(x) = \frac{e^c}{(n+1)!} x^{n+1}$. $x > 0 \Rightarrow c < x \Rightarrow e^c < e^x$

et $|R_n(x)| = \left|\frac{e^c x^{n+1}}{(n+1)!}\right| = \frac{e^c |x^{n+1}|}{(n+1)!} \leq \frac{e^x x^{n+1}}{(n+1)!}$

Si $x<0 \Rightarrow c<0$ et $e^c<1$ donc

$$|R_n(x)| = \frac{e^c}{(n+1)!}|x|^{n+1} \leq \frac{|x|^{n+1}}{(n+1)!}$$

En tous cas:

$$|R_n(x)| \leq \frac{e^{|x|}|x|^{n+1}}{(n+1)!}$$

EXERCICE 3

$f(x) = \frac{1}{1-x} \qquad x<1.$

$\frac{1}{1-x} = 1+x+x^2+ \quad +x^n+ \ldots \qquad \forall\ |x|<1$

$P_n(x) = 1+x+\ldots+x^n = \frac{1-x^{n+1}}{1-x}$

$f(x) - P_n(x) = \frac{1}{1-x} - \frac{1-x^{n+1}}{1-x} = \frac{x^{n+1}}{1-x} = R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$

$\Rightarrow \frac{x^{n+1}}{1-x} = \frac{\cancel{(n+1)!}}{\cancel{(n+1)!}(1-c)^{n+2}}x^{n+1} \Leftrightarrow$

$\Leftrightarrow \frac{x^{n+1}}{1-x} = \frac{x^{n+1}}{(1-c)^{n+2}} \Leftrightarrow x^{n+1}\left(\frac{1}{1-x} - \frac{1}{(1-c)^{n+2}}\right) = 0 \Leftrightarrow$

$x=0$ ou $\frac{1}{1-x} = \frac{1}{(1-c)^{n+2}} \Leftrightarrow (1-c)^{n+2} = 1-x \Leftrightarrow$

$\Leftrightarrow 1-c = \sqrt[n+2]{1-x} \Leftrightarrow 1-\sqrt[n+2]{1-x} = c$

EXERCICE 4

1\. $f(x) = \cosh(x)$

$f'(x) = \sinh(x)$

$f''(x) = \cosh(x)$

$f'''(x) = \sinh(x)$

$\vdots$

$$f^{(n)}(x) = \begin{cases} \sinh(x) & n=2k+1 \\ \cosh(x) & n=2k \end{cases}$$

récurrence

$n=1 \quad f'(x) = \sinh(x)$ ✓

Hypothèse: $n=2k$ $\quad f^{(2k)}(x)=\cosh(x)$

A voir: $n=2k+1$ $\quad f^{(2k+1)}(x)=(f^{(2k)})'=(\cosh(x))'=\sinh(x)$ ✓

b) $a_n=\dfrac{f^{(n)}(0)}{n!}$

$a_0=f(0)=\cosh(0)=1$ $\qquad a_2=\dfrac{f''(0)}{2!}=\dfrac{1}{2}$

$a_1=\dfrac{f'(0)}{1!}=\sinh(0)=0$ $\qquad a_3=\dfrac{f'''(0)}{3!}=0$

$$a_n=\begin{cases} n=2k+1 & 0 \\ n=2k & \dfrac{1}{n!}=\dfrac{1}{(2k)!}\end{cases}$$

c) $P_n(x)=a_0+a_1x+a_2x^2+\quad+a_nx^n$

$$=1+0+\frac{1}{2!}x^2+0+\frac{1}{4!}x^4+\quad+\frac{1}{(2k)!}x^{2k}$$

Le reste est $R_n(x)=\dfrac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$ $\quad c\in[0,x]$ où $n=2k$

$$R_{2k}=\frac{f^{(2k+1)}(c)}{(2k+1)!}x^{2k+1}$$

En tous cas $|f^{(2k+1)}(c)|\leq 1$ $\quad(\cosh(c)$ ou $\sinh(c))$

car $|f^{(2k+1)}(c)|\leq|\cosh(c)|$ car $|\sinh(c)|<\cosh(c)|$

Alors

$$0\leq|R_n(x)|=\left|\frac{f^{(2k+1)}(c)}{(2k+1)!}x^{2k+1}\right|\leq\left|\frac{\cosh(c)}{(2k+1)!}x^{2k+1}\right|<$$

$$<\left|\frac{\cosh(x)}{(2k+1)!}x^{2k+1}\right|$$

$$\lim_{k\to+\infty}\left|\frac{\cosh(x)}{(2k+1)!}x^{2k+1}\right|=(\cosh(x))\cdot\lim_{k\to+\infty}\frac{|x|^{2k+1}}{(2k+1)!}=\cosh(x)\cdot 0=0$$

car ! $\boxed{\lim_{n\to+\infty}\frac{x^n}{n!}=0}$ $\quad\forall x\in\mathbb{R}$

Alors $\lim_{n\to+\infty}|R_n(x)|=0$ $\quad\underline{\forall x\in\mathbb{R}}$

2. $f(x) = x \cdot e^x$

a) $f'(x) = e^x + xe^x = e^x(x+1)$
$f''(x) = e^x + e^x + xe^x = e^x(x+2)$.
$f'''(x) = e^x(x+3)$

$f^{(n)}(x) = e^x(x+n)$

récurrence

$n=1$ $f'(x) = e^x(x+1)$ ✓

$n=k$ On suppose $f^{(k)}(x) = e^x(x+k)$

$n=k+1$ 'A voir $f^{(k+1)}(x) = e^x(x+k+1)$

$f^{(k+1)}(x) = (f^{(k)}(x))' = (e^x(x+k))' = e^x(x+k) + e^x \cdot 1 =$
$= e^x(x+k+1)$ ✓

b) $a_n = \dfrac{f^{(n)}(0)}{n!}$

$a_0 = \dfrac{f(0)}{0!} = 0$ $a_2 = \dfrac{f''(0)}{2!} = \dfrac{2}{2} = 1$

$a_1 = \dfrac{f'(0)}{1!} = 1$ $a_3 = \dfrac{f'''(0)}{3!} = \dfrac{3}{6} = \dfrac{1}{2}$

$a_4 = \dfrac{f^{(4)}(0)}{4!} = \dfrac{4}{6\cdot 4} = \dfrac{1}{6}$ $a_5 = \dfrac{f^{(5)}(0)}{5!} = \dfrac{\cancel{5}}{6\cdot 4\cdot \cancel{5}} = \dfrac{1}{24}$

$a_n = \dfrac{1}{(n-1)!}$

c) $P(x) = 0 + 1\cdot x + 1\cdot x^2 + \frac{1}{2}x^3 + \ldots + \frac{1}{(n-1)!}x^n + R_n(x)$

Où $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} = \dfrac{e^c(c+n+1)}{(n+1)!}x^{n+1} \to 0$
$\forall x \in \mathbb{R}$

*. On peut aussi écrire la série de Taylor

$$x e^x = x \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$$

et comme $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ converge $\forall x$
aussi $\sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$ converge aussi

3) $f(x) = \frac{1}{1+x}$

a) $f'(x) = -\frac{1}{(1+x)^2}$ $\qquad f''(x) = \frac{2}{(x+1)^3}$

$f'''(x) = -\frac{6}{(x+1)^4}$ $\qquad f^{(4)}(x) = \frac{24}{(x+1)^5}$

$$f^{(n)}(x) = \begin{cases} -\frac{n!}{(x+1)^{n+1}} & n = 2k+1 \\ \frac{n!}{(x+1)^{n+1}} & n = 2k \end{cases} \Rightarrow f^{(n)}(x) = (-1)^n \frac{n!}{(x+1)^{n+1}}$$

recurrence

$n = 1 \qquad f'(x) = (-1)^1 \cdot \frac{1!}{(x+1)^2}$ ✓

$n = k \qquad f^{(k)}(x) = (-1)^k \frac{k!}{(1+x)^{k+1}}$ Hypothèse

$n = k+1 \qquad f^{(k+1)}(x) = (-1)^k \cdot k! \, (-k-1)(1+x)^{-k-2} =$
$= \frac{(-1)^{k+1} (k+1)!}{(1+x)^{k+2}}$ ✓

b) $a_0 = f(0) = 1$ $\qquad a_3 = \frac{f'''(0)}{3!} = -\frac{6}{6} = -1$

$a_1 = \frac{f'(0)}{1} = -1$ $\qquad a_4 = 1$

$a_2 = \frac{f''(0)}{2!} = \frac{2}{2} = 1$

$\underline{a_n = (-1)^n}$

c) $f(x) = 1 - x + x^2 - x^3 + x^4 - x^5 + \dots - x^{2k+1}$

$n = 2k+1 \qquad R_{2k+1}(x) = \dfrac{f^{(2k+2)}(c)}{(2k+3)!}\, x^{2k+3} \Rightarrow$

$\left|R_{2k+1}(x)\right| = \left|\dfrac{(-1)^{2k+2}(2k+2)!}{(1+c)^{2k+3}} \cdot \dfrac{x^{2k+3}}{(2k+3)!}\right| \leq |x|^{2k+3} \to 0$

si $\boxed{|x| < 1}$

4. $f(x) = \ln(1+x)$

a) $f'(x) = \dfrac{1}{1+x} \qquad f'''(x) = \dfrac{2}{(x+1)^3}$

$f''(x) = -\dfrac{1}{(1+x)^2}$

$f^{(n)}(x) = (-1)^{n+1}\dfrac{(n-1)!}{(1+x)^n}$

b) $a_0 = \dfrac{f(0)}{0!} = 0 \qquad a_3 = \dfrac{f'''(0)}{3!} = \dfrac{2}{6} = \dfrac{1}{3}$

$a_1 = \dfrac{f'(0)}{1!} = +1 \qquad a_4 = -\dfrac{1}{4}$

$a_2 = \dfrac{f''(0)}{2!} = \dfrac{-1}{2} \qquad a_5 = \dfrac{1}{5}$

c) $f(x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \dfrac{x^5}{5} - \dots = \sum_{n=1}^{\infty} (-1)^{n+1}\dfrac{x^n}{n}$

$\left|R_n(x)\right| = \left|\dfrac{n!}{(1+c)^{n+1}} \cdot \dfrac{x^{n+1}}{(n+1)!}\right| \leq |x|^{n+1} \to 0$ pour $\underbrace{|x| < 1}_{}$

comme $n \to +\infty$

Exercice 5

a) $f(x) = \sqrt{1+x^2} =$

$$P_3(x) = 1 + \frac{x^2}{2} - \frac{x^4}{8}$$

b) $f(x) = \tan(x)$

$$P_3(x) = x + \frac{x^3}{3} + \frac{2x^5}{15}$$

Exercice 6

a) $$y = \frac{\sin(x) - x\cos(x)}{x - \sin x} \simeq \frac{\frac{x^3}{3} - \frac{x^5}{30}}{\frac{x^3}{6} - \frac{x^5}{120}} =$$

$$= \frac{4\,\cancel{x^3}(10-x^2)}{\cancel{x^3}(20-x^2)}$$

$$\lim_{x\to 0} y = \lim_{x\to 0} \frac{4(10-x^2)}{20-x^2} = \frac{4\cdot 10}{20} = 2$$

b) $$y = \frac{\tan(x) - \sin(x)}{x^3} \simeq \frac{\frac{x^3}{2} + \frac{x^5}{8}}{x^3} = \frac{\frac{4x^3+x^5}{8}}{x^3} =$$

$$= \frac{4x^3+x^5}{8x^3} = \frac{\cancel{x^3}(4+x^2)}{8\cancel{x^3}} = \frac{4+x^2}{8}$$

$$\lim_{x\to 0} y = \lim_{x\to 0} \frac{4+x^2}{8} = \frac{1}{2}$$