LDDR - Niveau II : Analyse Combinatoire Solutions.

EXERCICE 12

a) 3 possibilités pour la 1re place
2 —//— 2me —//—
1 —//— —//— 3me —//—

Donc $3 \cdot 2 \cdot 1 = 3!$

b) $4!$

c) $12!$

EXERCICE 13

$$\sum_{i=0}^{n} i\,(i!) = (n+1)! - 1$$

$n=0$ $\quad 0\cdot(0!) = 0 = 1! - 1 = 0$ ✓

$n=k$ $\quad \sum_{i=0}^{k} i\,(i!) = (k+1)! - 1$ Hyp

$n=k+1$ A voir. $\sum_{i=0}^{k+1} i\,(i!) = (k+2)! - 1$

$$\sum_{i=0}^{k+1} i\,(i!) = \sum_{i=0}^{k} i\,(i!) = (k+1)! - 1 + (k+1)(k+1)! =$$

$$= (k+1)!\,(1+k+1) - 1 = (k+1)!\,(k+2) - 1 = (k+2)! - 1$$

EXERCICE 14

$$(a+b)^4 = \sum_{k=0}^{4} \binom{4}{k} \cdot a^{4-k} \cdot b^k =$$

$$\binom{4}{0} a^4 \cdot b^0 + \binom{4}{1} a^3 \cdot b + \binom{4}{2} a^2 b^2 + \binom{4}{3} a\, b^3 + \binom{4}{4} a^0 b^4$$

$$= a^4 + 4a^3 b + 6a^2 b^2 + 4ab^3 + b^4$$

$$(2x-1)^5 = \sum_{k=0}^{5} \binom{5}{k} (2x)^{5-k} \cdot (-1)^k =$$

$$= \binom{5}{0}(2x)^5(-1)^0 + \binom{5}{1}(2x)^4(-1) + \binom{5}{2}(2x)^3(-1)^2 + \binom{5}{3}(2x)^2(-1)^3 + \binom{5}{4}(2x)(-1)^4 +$$

$$+ \binom{5}{5}(2x)^0(-1)^5 =$$

$$= 32x^5 - 5\cdot 16x^4 + 10\cdot 8x^3 - 10\cdot 4x^2 + 5\cdot 2x - 1 =$$

$$= 32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1$$

EXERCICE 15

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \qquad \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$\binom{n-1}{k-1} = \frac{(n-1)!}{(k-1)!(n-1-k+1)!} = \frac{(n-1)!}{(k-1)!(n-k)!}$$

$$\binom{n-1}{k} = \frac{(n-1)!}{k!(n-k-1)!}$$

$$\frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} = \frac{k(n-1)!}{k!(n-k)!} + \frac{(n-1)!(n-k)}{k!(n-k)!} = \frac{(n-1)!(k+n-k)}{k!(n-k)!} =$$

$$= \frac{n!}{k!(n-k)!}$$

EXERCICE 16

A prouver: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \quad n \in \mathbb{N}^*$

Ancrage: $n=1 \quad \sum_{k=0}^{1} \binom{1}{k} a^k b^{1-k} = \binom{1}{0} a^0 b^{1-0} + \binom{1}{1} a^1 b^{1-1} = b + a$

Hypothèse: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

$\ell = k+1 \Rightarrow k = \ell - 1$

$\underline{n+1}$ $\quad (a+b)^{n+1} = (a+b)^n (a+b) \overset{Hyp}{=} (a+b)\left(\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}\right) =$

$= a\left(\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}\right) + b\left(\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}\right) = \sum_{k=0}^{n} \binom{n}{k} a^{k+1} b^{n-k} + \sum_{k=0}^{n} \binom{n}{k} a^k b^{n+1-k}$

$= \sum_{\ell=1}^{n+1} \binom{n}{\ell-1} a^\ell b^{n+1-\ell} + \sum_{k=0}^{n} \binom{n}{k} a^k b^{n+1-k} =$

$= \sum_{\ell=1}^{n} \binom{n}{\ell-1} a^\ell b^{n+1-\ell} + \binom{n}{n+1-1} a^{n+1} b^{n+1-(n+1)} + \sum_{k=1}^{n} \binom{n}{k} a^k b^{n+1-k} + \binom{n}{0} a^0 b^{n+1-0} =$

$= \sum_{\ell=1}^{n} \binom{n}{\ell-1} a^\ell b^{n+1-\ell} + \sum_{k=1}^{n} \binom{n}{k} a^k b^{n+1-k} + \binom{n}{n} a^{n+1} b^0 + \binom{n}{0} a^0 b^{n+1} \overset{k=\ell}{=}$

$= \sum_{\ell=1}^{n} \binom{n}{\ell-1} a^\ell b^{n+1-\ell} + \sum_{\ell=1}^{n} \binom{n}{\ell} a^\ell b^{n+1-\ell} + \binom{n}{n} a^{n+1} b^0 + \binom{n}{0} a^0 b^{n+1} =$

$= \sum_{\ell=1}^{n} \left(\binom{n}{\ell-1} + \binom{n}{\ell}\right) a^\ell b^{n+1-\ell} + \binom{n}{n} a^{n+1} b^0 + \binom{n}{0} a^0 b^{n+1}$

D'après Ex 15: $\binom{m}{\ell} = \binom{m-1}{\ell-1} + \binom{m-1}{\ell}$. En posant $m-1 = n \Rightarrow m = n+1$

on obtient: $\binom{n+1}{\ell} = \binom{n}{\ell-1} + \binom{n}{\ell}$. Ainsi

$(a+b)^{n+1} = \sum_{\ell=1}^{n} \binom{n+1}{\ell} a^\ell b^{n+1-\ell} + \underbrace{\binom{n}{n}}_{1} a^{n+1} b^0 + \binom{n}{0} a^0 b^{n+1}$

et $\binom{n}{n} = 1 = \binom{n+1}{n+1} = \binom{n}{0} = \binom{n+1}{0}$

Par conséquent:

$$(a+b)^{n+1} = \sum_{\ell=1}^{n} \binom{n+1}{\ell} a^\ell b^{n+1-\ell} + \binom{n+1}{n+1} a^{n+1} b^0 + \binom{n+1}{0} a b^{n+1} =$$

$$= \sum_{\ell=0}^{n+1} \binom{n+1}{\ell} a^\ell b^{n+1-\ell}$$

EXERCICE 17

$$\binom{42}{6}\binom{6}{1} = 31474716$$

EXERCICE 18

$$\underbrace{\overset{3}{\square}\ \overset{3}{\square}\ \overset{3}{\square}\ \dots\ \overset{3}{\square}}_{13 \text{ matches}} = 3^{13} = 1594323$$

EXERCICE 19

$$2+4+1+6 = 13$$

$$\frac{13!}{2!\,4!\,1!\,6!} = 1037836800$$

EXERCICE 20

$$\binom{36}{9}\cdot\binom{27}{9}\cdot\binom{18}{9}\cdot\binom{9}{9} \approx 2.1 \cdot 10^{19}$$

EXERCICE 21

On suppose que la place (l'odre) joue de rôle

a) $\underset{12}{\square}\ \underset{11}{\square}\ \underset{10}{\square}\ \underset{9}{\square} = 11880$

b) $\square\ \square\ \square\ \square$

1ère cas	0	tableau		1
2	-//-	1	-//-	$\binom{12}{1}\cdot 4$
3	-//-	2	-//-	$\binom{12}{2}\cdot\frac{4!}{2!} = \binom{12}{2}\cdot 3\cdot 4$
4	-//-	3	-//-	$\binom{12}{3}\cdot\frac{4!}{1!} = \binom{12}{3}\cdot 4\cdot 3\cdot 2$
5	-//-	4	-//-	$\binom{12}{4}\cdot\frac{4!}{0!} = \binom{12}{4}\cdot 4!$

+

EXERCICE 22

$1 \cdot 5!\, 4! = 2880$

EXERCICE 23

a) 6 M 3 C

$6 + 1 = 7 \qquad 7!\, 3! = 30240$

b) 6 A 5 F 2 M

$3!\, 6!\, 5!\, 2! = 172806$

EXERCICE 24

a)

$$\underset{5}{\square} \cdot \underset{4}{\square} \cdot \underset{3}{\square} \cdot \underset{2}{\square} = 120$$

$$\boxed{2} \quad \underset{4}{\square} \quad \underset{3}{\square} \quad \underset{2}{\square} = 36$$

$$\underset{2}{\square} \cdot \underset{3}{\square} \quad \boxed{2} \quad \boxed{5} = 6$$

b)

$$\underset{5}{\square} \cdot \underset{5}{\square} \cdot \underset{5}{\square} \cdot \underset{5}{\square} = 5^4 = 625$$

$$\boxed{2} \quad \underset{5}{\square} \cdot \underset{5}{\square} \cdot \underset{5}{\square} = 5^3 = 125$$

$$\underset{5}{\square} \cdot \underset{5}{\square} \quad \boxed{2} \quad \boxed{5} = 25$$