EXERCICE 1.42

1) $f(x) = x^3 - 3x$

$D_f = \mathbb{R}$

$\cap Ox$: $f(x) = 0 \Leftrightarrow x(x^2-3) = 0 \Leftrightarrow x_1 = 0 \quad x_2 = \sqrt{3} \quad x_3 = -\sqrt{3}$

$I_1(0;0) \quad I_2(\sqrt{3};0) \quad I_3(-\sqrt{3};0)$

$\cap Oy$: $x = 0 \quad y = 0 \quad I_y(0,0)$

TS

x	$-\infty$		$-\sqrt{3}$		0		$\sqrt{3}$		$+\infty$
x		$-$		$-$	0	$+$		$+$	
x^2-3		$+$	0	$-$		$-$	0	$+$	
f		$-$	0	$+$	0	$-$	0	$+$	

AV $\emptyset$ AH/AO $\emptyset$

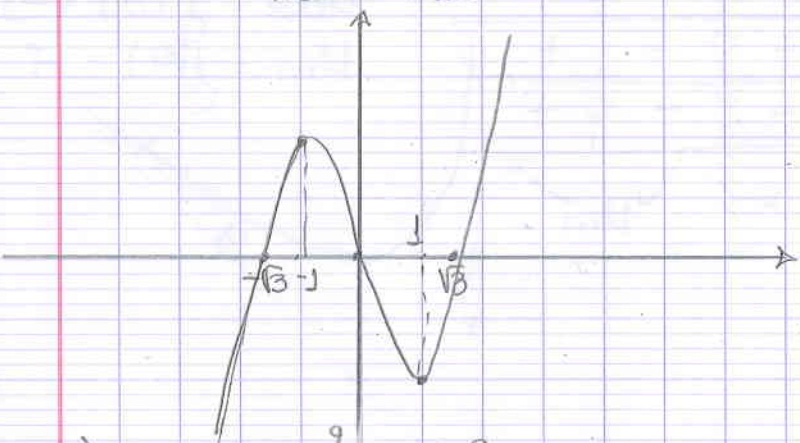
TV $f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$

$f'(x) = 0 \Leftrightarrow x_1 = 1 \quad x_2 = -1$

x	$-\infty$		-1		1		$+\infty$
$f'(x) = 3x^2-3$		$+$	0	$-$	0	$+$	
f		$\nearrow$	Max	$\searrow$	Min	$\nearrow$	

$Max = f(-1) = 2$

$Min = f(1) = -2$



2) $f(x) = \dfrac{x^2 - x + 2}{x-2}$ $\quad D_f = \mathbb{R} \setminus \{2\}$

$\cap Ox$: $f(x) = 0 \Leftrightarrow x^2 - x + 2 = 0 \quad \Delta = 1 - 8 = -7 < 0$ Imp.

$\cap Oy$: $x = 0 \quad y = \frac{2}{-2} = -1 \quad I_y(0;-1)$

TS

x	$-\infty$		2		$+\infty$
x^2-x+2		$+$	0	$+$	
$x-2$		$-$	0	$+$	
f		$-$	$\|\|$	$+$	

AV $\quad \lim_{x\to 2} f(x) = \left[\frac{4}{0}\right] = \begin{cases} \lim_{x\to 2^+} f(x) = +\infty \\ \lim_{x\to 2^-} f(x) = -\infty \end{cases}$ $\quad \underline{x=2}$ AV

AO/AH

$$\begin{array}{r|l} x^2 - x + 2 & x-2 \\ \hline -x^2 + 2x & x+1 \\ \hline x + 2 & \\ -x + 2 & \\ \hline 4 & \end{array}$$

AO $\quad \underline{y = x+1}$

$$f'(x) = \frac{(2x-1)(x-2) - (x^2 - x + 2)}{(x-2)^2} =$$

$$= \frac{2x^2 - \cancel{x} - 4x + \cancel{2} - x^2 + \cancel{x} - \cancel{2}}{(x-2)^2} = \frac{x^2 - 4x}{(x-2)^2}$$

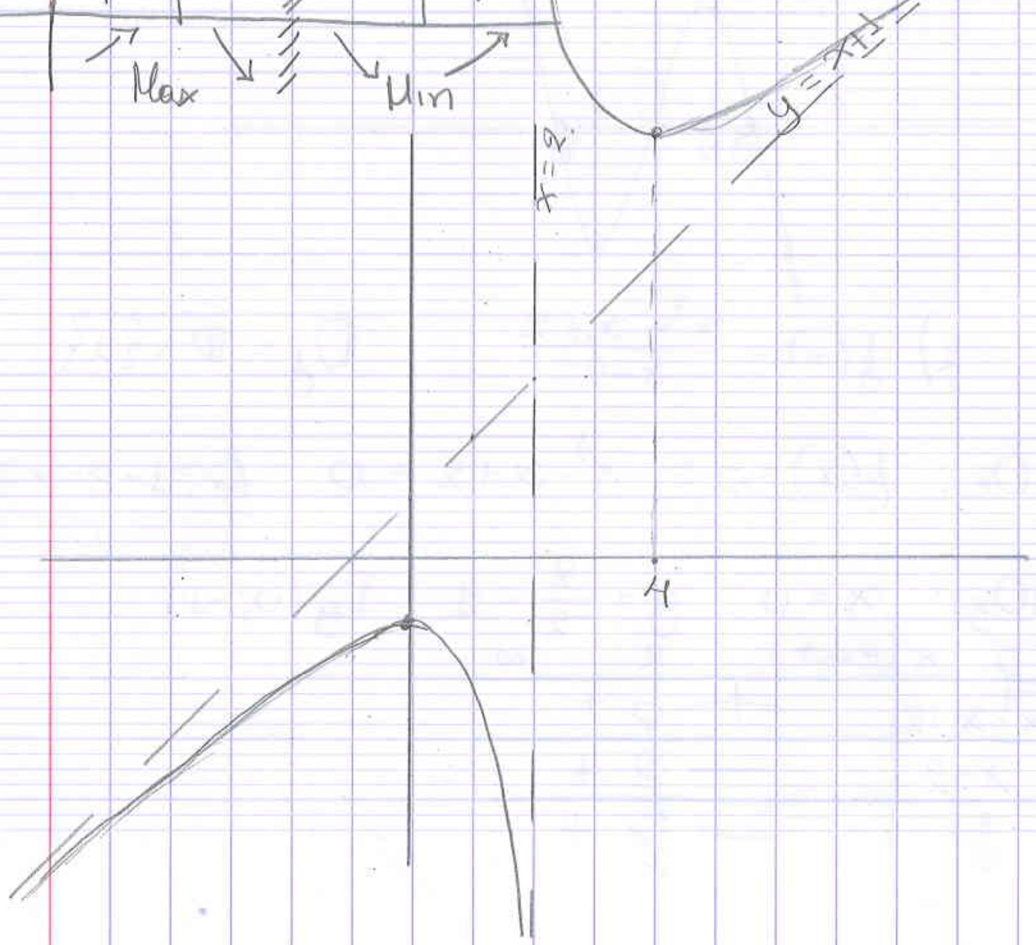
$$= \frac{x(x-4)}{(x-2)^2}$$

$$f'(x) = 0 \Leftrightarrow x(x-4) = 0 \Leftrightarrow x = 0 \quad x = 4$$

TV	$-\infty$		0		2		4		$+\infty$
x^2-4x		$+$	0	$-$	|	$-$	0	$+$	
$(x-2)^2$		$+$		$+$	0	$+$		$+$	
f'		$+$	0	$-$	||	$-$	0	$+$	
f		$\nearrow$	Max	$\searrow$	||	$\searrow$	Min	$\nearrow$	

$Max = f(0) = -1$

$Min = f(4) = 7$



3) $f(x) = \sqrt{9-x^2}$ $\qquad 9-x^2 \geq 0 \Leftrightarrow (3-x)(3+x) \geq 0$

	$-\infty$		-3		3		$+\infty$
$9-x^2$		$-$	0	$+$	0	$-$	

$D_f = [-3;3]$

$\cap Ox$: $f(x)=0 \Leftrightarrow \sqrt{9-x^2}=0 \Leftrightarrow 9-x^2=0 \Leftrightarrow x=3 \quad x=-3$

$I_{1x}(3;0) \quad I_{2x}(-3;0)$

$\cap Oy$: $x=0 \quad y=3 \quad I_y(0;3)$

TS

x	-3		$+3$
f	0	$+$	0

AV: $\emptyset$ $\qquad$ AH/AO $\emptyset$

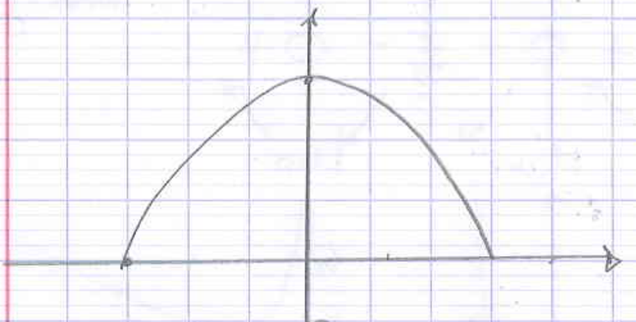
$$f'(x) = \frac{1}{2\sqrt{9-x^2}}(-2x) = \frac{-\cancel{2}x}{\cancel{2}\sqrt{9-x^2}} = -\frac{x}{\sqrt{9-x^2}}$$

$f'(x)=0 \Leftrightarrow x=0$

TV

	-3		0		3
f'		$+$	0	$-$	
f		$\nearrow$	Max	$\searrow$	

$Max = f(0) = 3$



4) $f(x) = \dfrac{(x+1)^3}{x^2-7x+10}$ $\qquad x^2-7x+10=0 \Leftrightarrow x_1=5 \quad x_2=2$

$D_f = \mathbb{R} \setminus \{2;5\}$

$\cap Ox$: $f(x)=0 \Leftrightarrow (x+1)^3=0 \Leftrightarrow x=-1 \qquad I_x(-1;0)$

$\cap Oy$: $x=0 \quad y=\frac{1}{10} \qquad I_y(0;\frac{1}{10})$

TS

	$-\infty$		-1		2		5		$+\infty$
$(x+1)^3$		$-$	0	$+$		$+$		$+$	
$x^2-7x+10$		$+$		$+$	0	$-$	0	$+$	
f		$-$	0	$+$	//	$-$	//	$+$	

$$f'(x) = \frac{3(x+1)^2(x^2-7x+10) - (x+1)^3(2x-7)}{(x^2-7x+10)^2} =$$

$$= \frac{(x+1)^2\left[3x^2-21x+30-(x+1)(2x-7)\right]}{(x^2-7x+10)^2}$$

$$= \frac{(x+1)^2\left[3x^2-21x+30-2x^2-2x+7x+7\right]}{(x^2-7x+10)^2}$$

$$= \frac{(x+1)^2(x^2-16x+37)}{(x^2-7x+10)^2}$$

$f'(x) = 0 \Leftrightarrow x+1=0 \Leftrightarrow x=-1$

$x^2-16x+37=0 \qquad \Delta = 108$

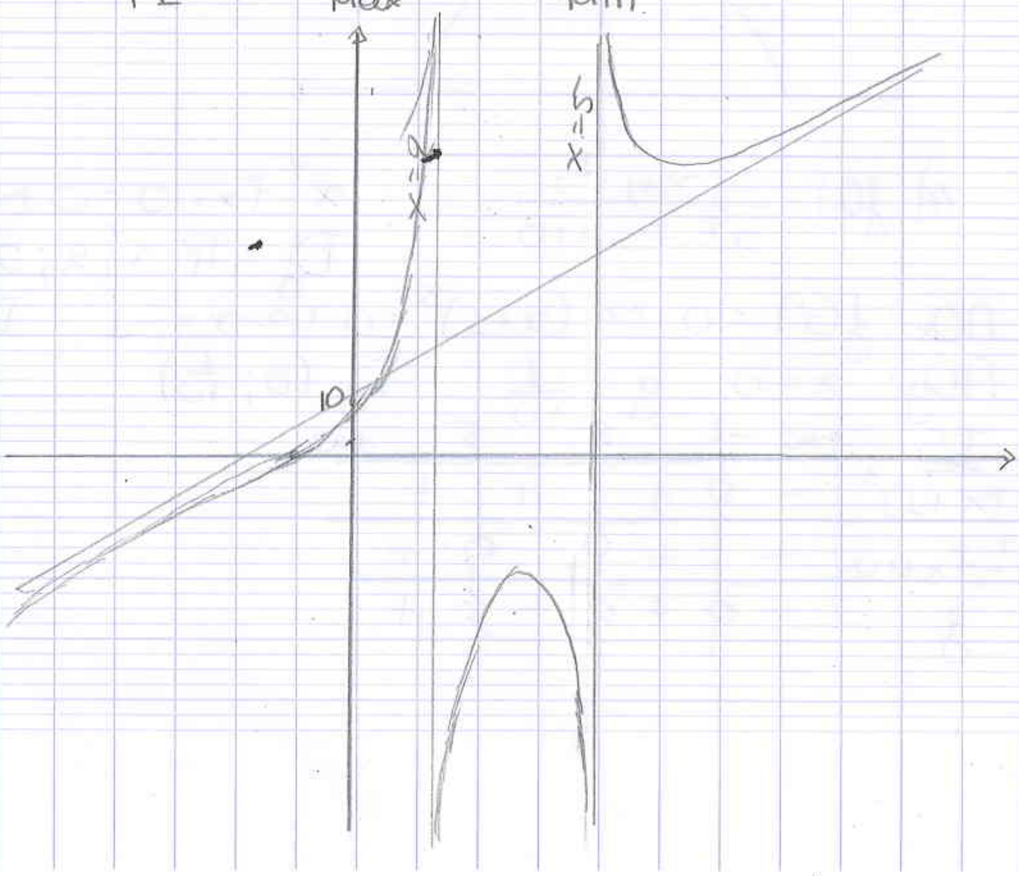
$$x_{1,2} = \frac{16 \pm \sqrt{108}}{2} = \frac{16 \pm 2\sqrt{27}}{2} = 8 \pm \sqrt{27}$$

$x_1 \approx 13.2 \qquad x_2 = 2.8$

TV x	$-\infty$		-1		2		2.8		5		13.2		$+\infty$
$(x+1)^2$		+	0	+	|	+	|	+	|	+	|	+	
$x^2-16x+37$		+	|	+	|	+	0	−	|	−	0	+	
$(x^2-7x+10)^2$		+	|	+	0	+	|	+	0	+	|	+	
f'		+	0	+	||	+	0	−	||	−	0	+	
f		↗		↗	||	↗		↘	||	↘		↗	
			PI				Max				Min		

$Max = f(2.8; -31.2)$

$Min = f(13.2; 31.2)$



s) $f(x) = \dfrac{x^2}{x^2-4}$ $\qquad D_f = \mathbb{R} \setminus \{\pm 2\}$

$\cap Ox$: $f(x) = 0 \Leftrightarrow x = 0$ $\qquad I_x\,(0;0)$.

$\cap Oy$: $x = 0 \quad y = 0.$ $\qquad I_y\,(0;0)$

<u>TS</u>

x	$-\infty$		-2		0		2		$+\infty$
x^2		+		+	0	+		+	
x^2-4		+	0	−		−	0	+	
f		+	‖	−	0	−	‖	+	

AV : $\lim\limits_{x \to 2} f(x) = \left[\frac{4}{0}\right] = \infty$ $\qquad$ AV $\underline{x = 2}$

$\lim\limits_{x \to -2} f(x) = \left[\frac{4}{0}\right] = \infty$ $\qquad$ AV $\underline{x = -2}$

AH/AO $\lim\limits_{x \to +\infty} f(x) = 1 \implies$ AH : $\underline{y = 1}$

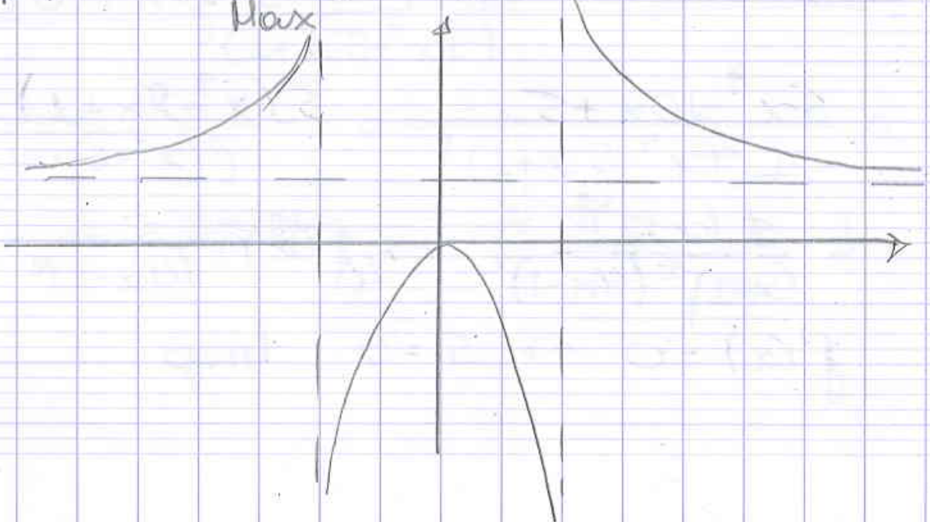
$$f'(x) = \frac{2x(x^2-4) - x^2(2x)}{(x^2-4)^2} = \frac{\cancel{2x^3} - 8x - \cancel{2x^3}}{(x^2-4)^2} =$$

$$= \frac{-8x}{(x^2-4)^2}$$

$f'(x) = 0 \Leftrightarrow x = 0$

<u>TV</u> $\qquad$ Max $= f(0) = 0$

x	$-\infty$		-2		0		2		$+\infty$
$-8x$		+		+	0	−		−	
$(x^2-4)^2$		+	0	+		+	0	+	
f'		+	‖	+	0	−	‖	−	
f		↗		↗	Max	↘		↘	



6) $f(x) = \dfrac{3x^2-5x+2}{4x^2-5x+1}$ $\qquad 4x^2-5x+1=0 \quad \Delta = 25-16=9$

$x_{1,2} = \dfrac{5 \pm 3}{8} = \begin{cases} 1 \\ \frac{1}{4} \end{cases}$

$D_f = \mathbb{R} \setminus \left\{1 ; \frac{1}{4}\right\}$

$\cap Ox$: $f(x)=0 \Leftrightarrow 3x^2-5x+2=0 \qquad \Delta = 25-24=1$

$I_{2x}\left(\frac{2}{3};0\right) \qquad x_{1,2} = \dfrac{5 \pm 1}{6} = \begin{cases} 1 \\ \frac{2}{3} \end{cases}$ Imp.

$\cap Oy$: $x=0 \quad y=2 \quad I_y(0;2)$

TS

x	$-\infty$		$1/4$		$2/3$		1		$+\infty$
$3x^2-5x+2$		+	0	−	|	−	0	+	
$4x^2-5x+1$		+		+	0	−	0	+	
f		+	//	−	0	+	//	+	

AV $\lim\limits_{x \to 1} f(x) = \left[\frac{0}{0}\right] \overset{fact}{=} \lim\limits_{x \to 1} \dfrac{3\cancel{(x-1)}(x-\frac{2}{3})}{4\cancel{(x-1)}(x-\frac{1}{4})} = \dfrac{1}{3}$

Trou $\left(1; \frac{1}{3}\right)$

$\lim\limits_{x \to \frac{1}{4}} f(x) = \left[\frac{0{,}94}{0}\right] = \infty \qquad$ AV $\underline{\underline{x = 1/4}}$

AO/AH: $\lim\limits_{x \to \pm\infty} f(x) = \dfrac{3}{4} \Rightarrow$ AH: $\underline{\underline{y = \frac{3}{4}}}$

$f'(x) = \dfrac{(6x-5)(4x^2-5x+1)-(3x^2-5x+2)(8x-5)}{(4x^2-5x+1)^2} =$

$= \dfrac{\cancel{24x^3}-30x^2+6x-20x^2+25x-5-\cancel{24x^3}+40x^2-16x+15x^2-25x+10}{(4x^2-5x+1)^2}$

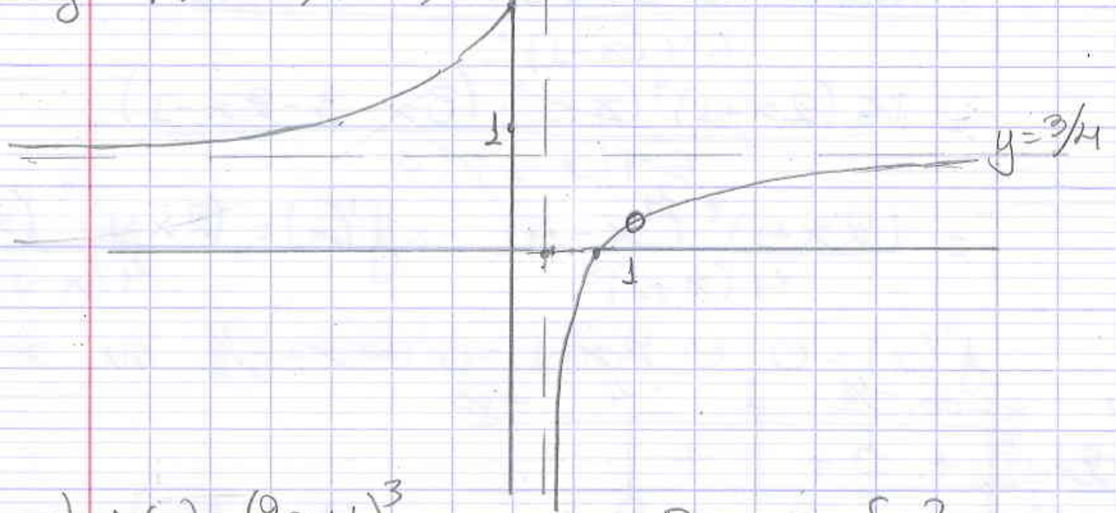
$= \dfrac{5x^2-10x+5}{(4x^2-5x+1)^2} = \dfrac{5(x^2-2x+1)}{(4x^2-5x+1)^2} = \dfrac{5(x-1)^2}{\left(4(x-1)(x-\frac{1}{4})\right)^2}$

$= \dfrac{5\cancel{(x-1)^2}}{\cancel{(x-1)^2}(4x-1)^2} \Rightarrow f'(x) = \dfrac{5}{(4x-1)^2}$

$f'(x) = 0 \Rightarrow 5 = 0$ Imp.

TV

x	$-\infty$		$1/4$		1		$+\infty$
f'		$+$	‖	$+$	‖	$+$	
f		↗	‖	↗	‖	↗	



7) $f(x) = \dfrac{(2x+1)^3}{8(x-1)^2}$ $\quad D_f = \mathbb{R} \setminus \{1\}$

$\cap Ox$: $f(x) = 0 \Leftrightarrow 2x+1 = 0 \Leftrightarrow x = -\frac{1}{2}$ $\quad I_x\left(-\frac{1}{2}; 0\right)$

$\cap Oy$: $x = 0 \quad y = \frac{1}{8}$ $\quad I_y\left(0; 1/8\right)$

TS

x	$-\infty$		$-1/2$		1		$+\infty$
$(2x+1)^3$		$-$	0	$+$	|	$+$	
$(x-1)^2$		$+$	|	$+$	0	$+$	
f		$-$	0	$+$	‖	$+$	

AV $\lim\limits_{x \to 1} f(x) = \left[\frac{27}{0^+}\right] = +\infty$ $\quad$ AV $\underline{\underline{x = 1}}$

AO $(2x+1)^3 = 8x^3 + 3 \cdot 4x^2 + 3 \cdot 2x + 1 = 8x^3 + 12x^2 + 6x + 1$

$8(x-1)^2 = 8x^2 - 16x + 8$

$$\begin{array}{r|l} 8x^3 + 12x^2 + 6x + 1 & 8x^2 - 16x + 8 \\ \cline{2-2} -\cancel{8x^3} + 16x^2 - 8x & x + \frac{7}{2} \\ \cline{1-1} \cancel{28x^2} - 2x + 1 & \\ -\cancel{28x^2} + 56x - 28 & \\ \cline{1-1} 54x - 27 & \end{array}$$

AO : $\underline{y = x + \frac{7}{2}}$

$$f'(x) = \frac{3(2x+1)^2 \cdot 2 \cdot 8 \cdot (x-1)^2 - (2x+1)^3 \cdot 16 \cdot (x-1)}{64(x-1)^4}$$

$$= \frac{16(2x+1)^2(x-1)\left[3(x-1)-(2x+1)\right]}{64(x-1)^4}$$

$$= \frac{\cancel{16}(2x+1)^2\cancel{(x-1)}(3x-3-2x-1)}{\cancel{64}\,4(x-1)^{\cancel{4}\,3}} =$$

$$= \frac{(2x+1)^2(x-4)}{4(x-1)^3} \Rightarrow f'(x) = \frac{(2x+1)^2(x-4)}{4(x-1)^3}$$

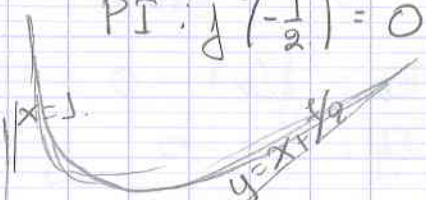
$f'(x) = 0 \Leftrightarrow 2x+1 = 0 \Leftrightarrow x = -\frac{1}{2}$ ou $x = 4$

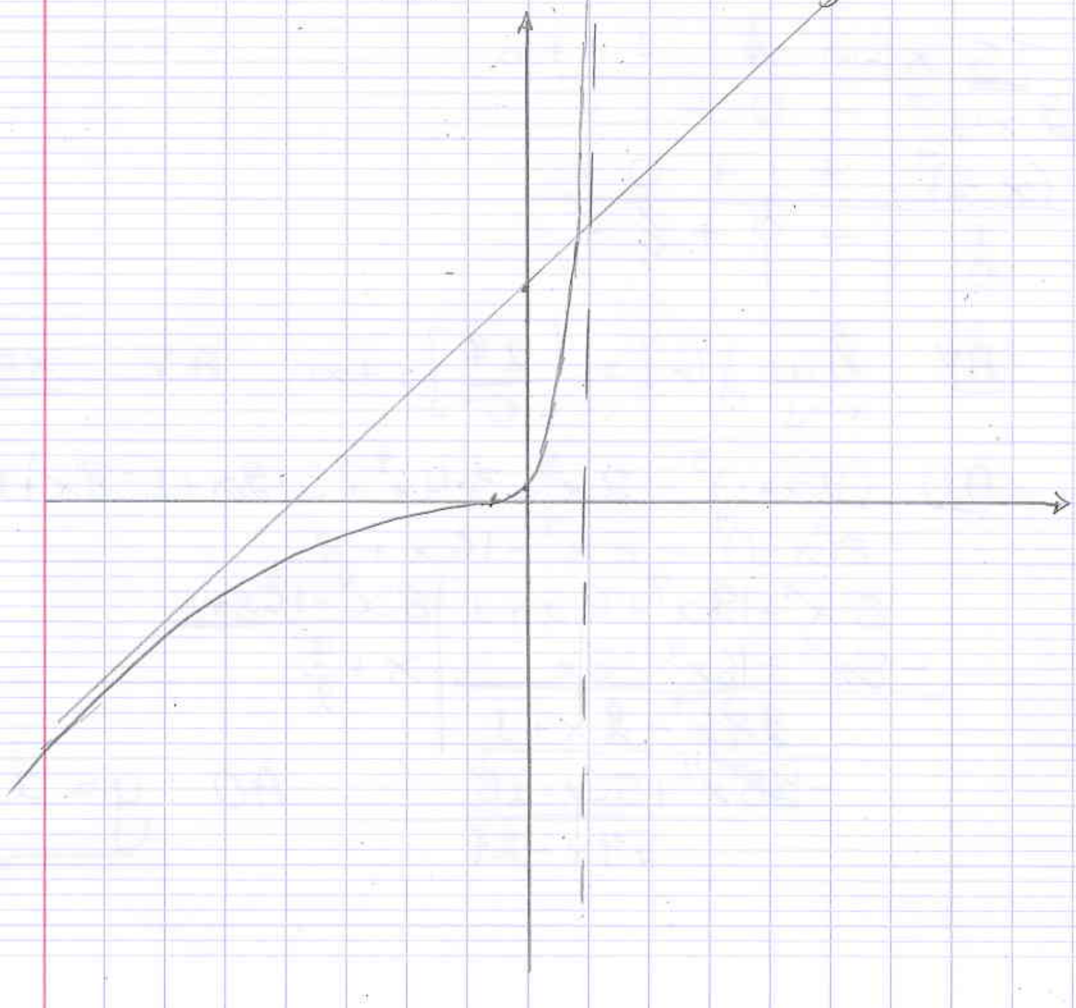
TV

x	$-\infty$	$-1/2$		1		4	$+\infty$
$(2x+1)^2$	$+$	0	$+$		$+$		$+$
$(x-4)$	$-$		$-$		$-$	0	$+$
$(x-1)^3$	$-$		$-$	0	$+$		$+$
f'	$+$	0	$+$	$\|$	$-$	0	$+$
f	$\nearrow$	PI	$\nearrow$	$\|$	$\searrow$	Min	$\nearrow$

Min $= f(4) = 10.13$

PI : $f\left(-\frac{1}{2}\right) = 0$





EXERCICE 1.43

$y = f(x) = 3\sin x - 4\cos x \qquad D_f = \mathbb{R}$

Parité $f(-x) = 3\sin(-x) - 4\cos(-x) = 3\sin x - 4\cos x$

Ni paire, ni impaire

Périodicité : $\sin(x+2\pi) = \sin x$ et $\cos(x+2\pi) = \cos x$

Donc $f(x+2\pi) = f(x) \Rightarrow T = 2\pi$

∩Ox $y = 0 \Leftrightarrow 3\sin x - 4\cos x = 0 \Leftrightarrow 3\sin x = 4\cos x$

$\Rightarrow \dfrac{\sin x}{\cos x} = \dfrac{4}{3} \Leftrightarrow \tan x = \dfrac{4}{3} \rightarrow \tan x = \tan(0.93)$

$\Rightarrow x = 0.93, \quad x = 0.93 + \pi = 4.07$ rad

$I_{1x}(0.93; 0) \quad I_{2x}(4{,}07; 0)$

∩Oy : $x = 0 \quad f(0) = -4 \qquad I_y(0; -4)$

TS :

x	0		0.93		4.07		2π
f		$-$	0	$+$	0	$-$	

AV $\varnothing$

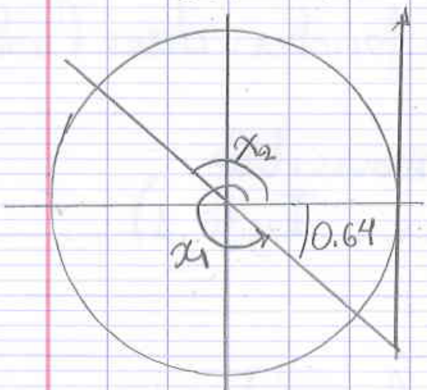
AH/AO $\varnothing$

Dérivée : $f'(x) = 3\cos x + 4\sin x$

PTH : $f'(x) = 0 \Leftrightarrow 3\cos x = -4\sin x \Leftrightarrow \tan x = -\dfrac{3}{4}$

$\Rightarrow x = \tan^{-1}\left(-\dfrac{3}{4}\right) = 2\pi - 0.64 \Rightarrow x_1 = 5.64$

ou $x = 5.64 - \pi \Rightarrow x_2 = 2.5$

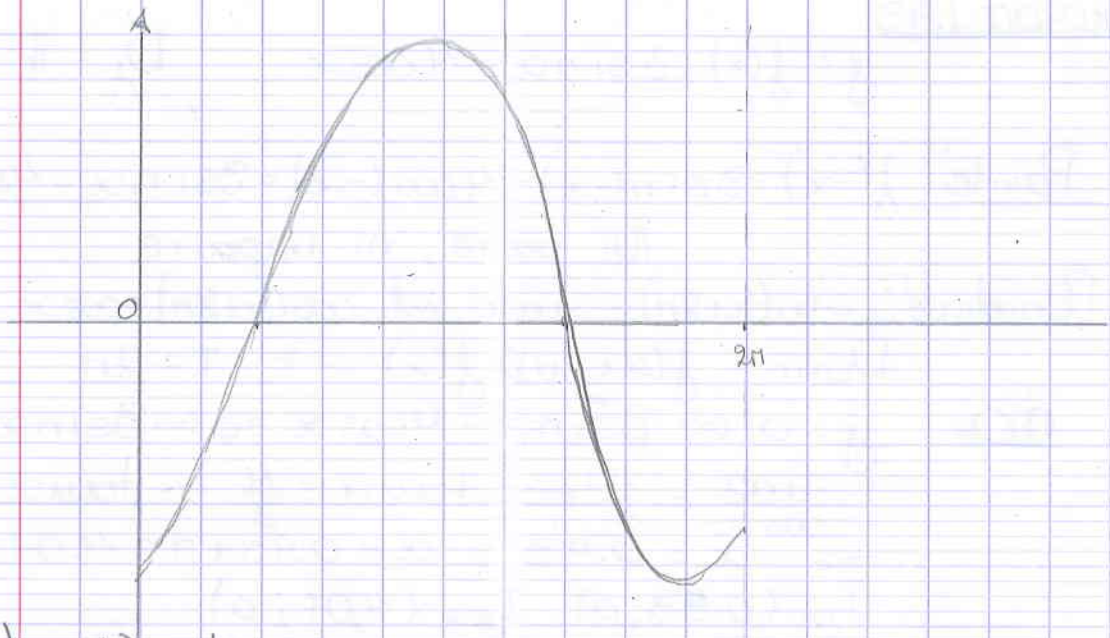


T.V

x	0		2.5		5.64		2π
f'		$+$	0	$-$	0	$+$	
f		↗	Max	↘	Min	↗	

Max $= f(2.5) = 5$

Min $= f(5.64) = -5$



2) $g(x) = \frac{1}{\cos(2x)}$

<u>Domaine</u> $\cos(2x) = 0 \Leftrightarrow 2x = \frac{\pi}{2} + 2k\pi \Leftrightarrow x = \frac{\pi}{4} + k\pi$

$2x = \frac{3\pi}{2} + 2k\pi \Leftrightarrow x = \frac{3\pi}{4} + k\pi$

$D_g = \mathbb{R} \setminus \left\{ \frac{\pi}{4} + 2k\pi, \frac{3\pi}{4} + k\pi, \ k \in \mathbb{Z} \right\}$

<u>Parité</u> $g(-x) = \frac{1}{\cos(-2x)} = \frac{1}{\cos 2x} = g(x)$ <u>Paire</u>

<u>Périodicité</u> $T = \frac{2\pi}{2} = \pi$: $g(x+\pi) = \frac{1}{\cos(2(x+\pi))} = \frac{1}{\cos(2x+2\pi)} = \frac{1}{\cos 2x}$

Alors on étude la fonction dans l'intervale $[0, \pi]$

$\cap Ox$: $g(x) = 0 \Leftrightarrow 1 = 0$ impossible

$\cap Oy$: $x = 0$ $g(0) = \frac{1}{\cos 0} = 1$ $I_y(0, 1)$

<u>TS</u>

x	0		$\pi/4$		$3\pi/4$		π
g		$+$	//	$-$	//	$+$	

AV $\lim\limits_{x \to \frac{\pi}{4}} g(x) = \left[\frac{1}{0}\right] <$ $\lim\limits_{x \to \frac{\pi}{4}^+} g(x) = -\infty$; $\lim\limits_{x \to \frac{\pi}{4}^-} g(x) = +\infty$ AV $\underline{\underline{x = \frac{\pi}{4}}}$

$\lim\limits_{x \to \frac{3\pi}{4}} g(x) = \left[\frac{1}{0}\right] =$ $\lim\limits_{x \to \frac{3\pi}{4}^+} g(x) = +\infty$; $\lim\limits_{x \to \frac{3\pi}{4}^-} g(x) = -\infty$ AV $\underline{\underline{x = \frac{3\pi}{4}}}$

AH/AO $\emptyset$

Dérivée $g'(x) = -\frac{1}{\cos^2(2x)} \cdot (-\sin 2x) \cdot 2 = \frac{2\sin 2x}{\cos^2(2x}$

PTH $g'(x) = 0 \Leftrightarrow \sin 2x = 0 \Leftrightarrow 2x = 0 \Leftrightarrow x = 0 \quad g(0) = 1.$

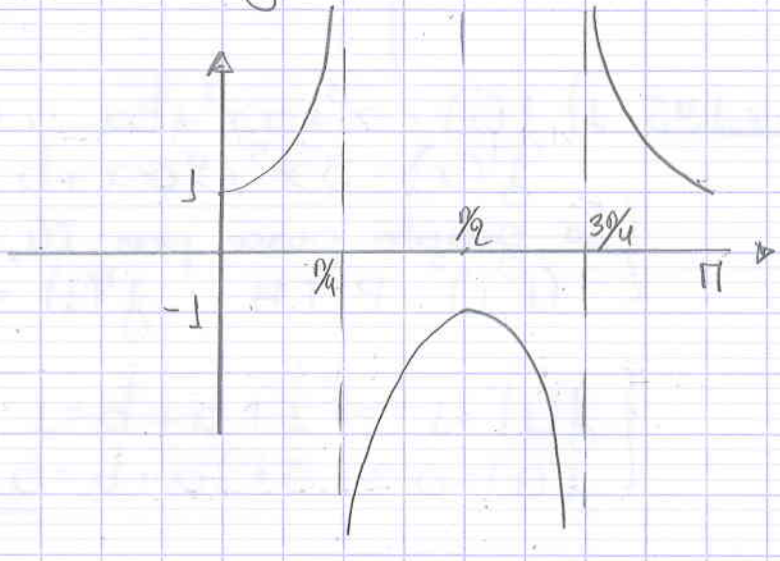
$2x = \pi \Leftrightarrow x = \frac{\pi}{2}. \quad g\left(\frac{\pi}{2}\right) = -1$

$2x = 2\pi \Leftrightarrow x = \pi \quad g(\pi) = 1$

TV

x	0		$\pi/4$		$\pi/2$		$3\pi/4$		π
$2\sin 2x$	0	$+$	‖	$+$	0	$-$	‖	$-$	0
$\cos^2(2x)$		$+$	‖	$+$		$+$	‖	$+$	
g'		$+$	‖	$+$	0	$-$	‖	$-$	
g	$g(0)=1$	$\nearrow$	‖	$\nearrow$	Max	$\searrow$	‖	$\searrow$	$g(\pi)=1$

$Max = g\left(\frac{\pi}{2}\right) = -1$



Exercice 1.44

1) $f(x) = \frac{x-3}{x+4} \qquad D_f = \mathbb{R} \setminus \{-4\}$

Dérivée $f'(x) = \frac{(x+4)-(x-3)}{(x+4)^2} = \frac{x+4-x+3}{(x+4)^2} = \frac{7}{(x+4)^2}$

PTH: $f'(x) = 0 \Leftrightarrow 7 = 0$ impossible

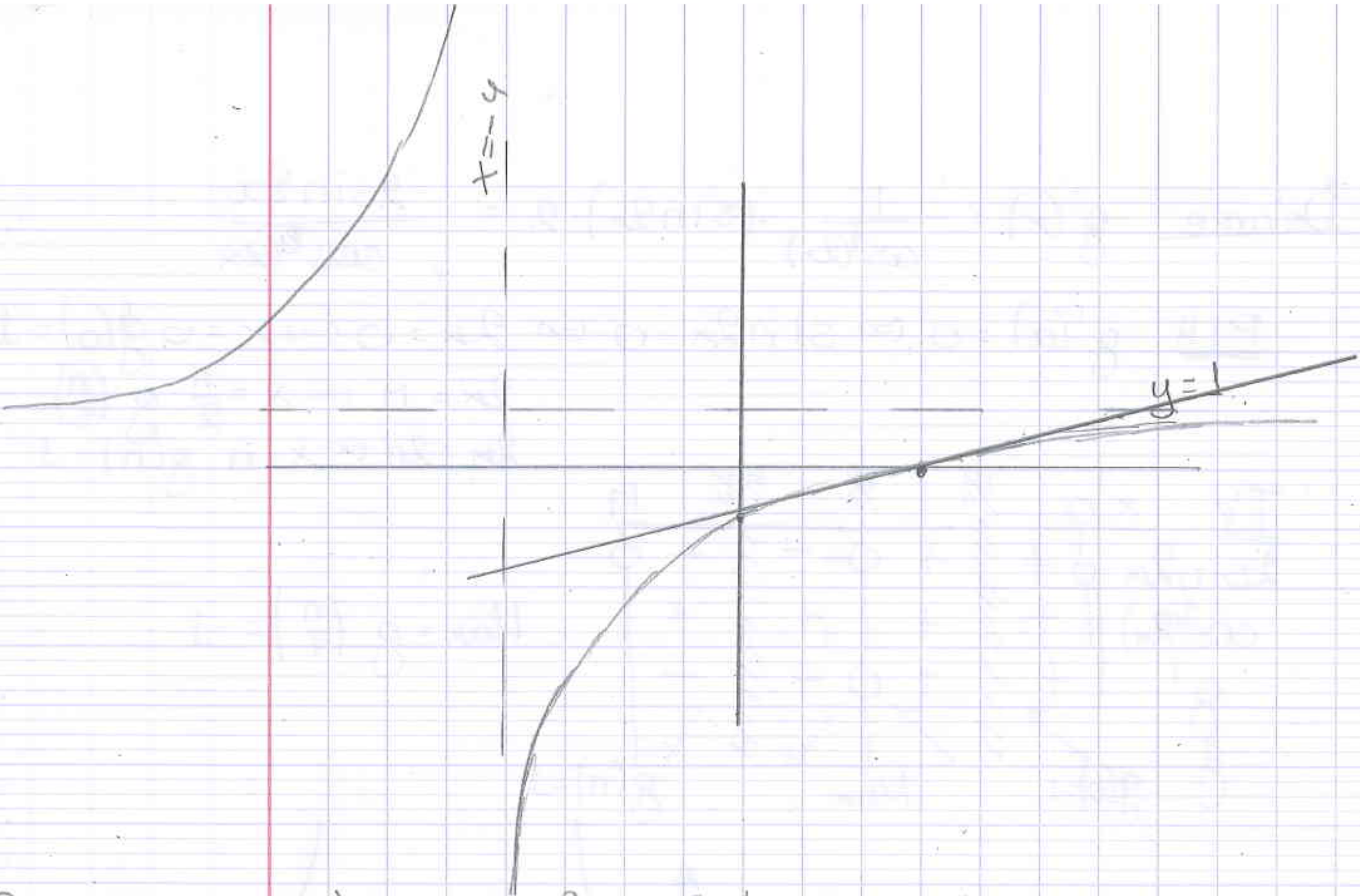
Donc il n' existe pas PTH

2) • $f(3) = 0 = y \qquad \cdot f'(3) = \frac{7}{49} = \frac{1}{7} = m$

$y = mx + h$

$0 = \frac{1}{7} \cdot 3 + h \Leftrightarrow h = -\frac{3}{7}$

$t: \; y = \frac{1}{7} x - \frac{3}{7}$



EXERCICE 1.45 1) $f(x) = x^3 + ax^2 + bx \quad a, b \in \mathbb{R}$

$f'(x) = 3x^2 + 2ax + b$

$\begin{cases} \text{Le graphe passe par } T(1;1) : f(1) = 1 \\ T(1,1) \text{ PTH } : f'(1) = 0 \end{cases}$

$\begin{cases} f(1) = 1 \Leftrightarrow 1 + a + b = 1 \Leftrightarrow a + b = 0 \\ f'(1) = 0 \Leftrightarrow 3 + 2a + b = 0 \Leftrightarrow 2a + b = -3 \end{cases}$

$-a = 3 \Leftrightarrow \underline{a = -3}$

$\underline{b = 3}$

$f(x) = x^3 - 3x^2 + 3x \qquad f'(x) = 3x^2 - 6x + 3$

PTH $\quad f'(x) = 0 \Leftrightarrow 3(x^2 - 2x + 1) = 0 \Leftrightarrow 3(x-1)^2 = 0$

$x = 1$

TV

x	$-\infty$		1		$+\infty$
f'		$+$	0	$+$	
f		$\nearrow$	PI	$\nearrow$	

$PI = f(1) = 1$

2) $f(x) = 2x^3 + ax^2 + bx - 5 \qquad t : y = -2.5x + 4$

Alors f passe par $(-1; 6.5)$

$x = -1$

$y = 6.5$

$f(-1) = 6.5 \Leftrightarrow 2(-1)^3 + a(-1)^2 + b(-1) - 5 = 6.5$
$\Leftrightarrow -2 + a - b - 5 = 6.5 \Leftrightarrow \underline{a - b = 13.5}$

On a en plus $f'(-1) = -2.5$
$f'(x) = 6x^2 + 2ax + b$
$f'(-1) = -2.5 \Leftrightarrow 6(-1)^2 + 2a(-1) + b = -2.5 \Leftrightarrow$
$\Leftrightarrow 6 - 2a + b = -2.5 \Leftrightarrow -2a + b = -8.5$
On a le système:
$$+\begin{cases} a - b = 13.5 \\ -2a + b = -8.5 \end{cases}$$
$-a = 5 \Leftrightarrow \underline{a = -5} \Rightarrow b = -5 - 13.5 = \underline{-18.5 = b}$

EXERCICE 1.46 1) $y = \dfrac{3x+2}{(x+1)^2}$ $y' = \dfrac{3(x+1)^2 - (3x+2)\cdot 2(x+1)}{(x+1)^4} =$

$= \dfrac{(x+1)(3x+3-6x-4)}{(x+1)^4} = \dfrac{\cancel{(x+1)}(-3x-1)}{(x+1)^{\cancel{4}3}} \Rightarrow$

$\Rightarrow y' = \dfrac{-3x-1}{(x+1)^3} = f'(x)$

- $f(1) = \dfrac{5}{4}$ $f'(1) = -\dfrac{1}{2}$

$\dfrac{5}{4} = -\dfrac{1}{2}\cdot 1 + h \Leftrightarrow h = \dfrac{5}{4} + \dfrac{1}{2} = \dfrac{5}{4} + \dfrac{2}{4} \Rightarrow$

$\Rightarrow h = \dfrac{7}{4}$: $t: y = -\dfrac{1}{2}x + \dfrac{7}{4}$

EXERCICE 1.47 $f(x) = \dfrac{3x(x-a)}{x^2+9} = \dfrac{3x^2 - 3ax}{x^2+9}$ $a \in \mathbb{R}^*$

$f'(x) = \dfrac{(6x-3a)(x^2+9) - (3x^2-3ax)\cdot 2x}{(x^2+9)^2}$

$= \dfrac{\cancel{6x^3} - 3ax^2 + 54x - 27a - \cancel{6x^3} + 6ax^2}{(x^2+9)^2}$

$= \dfrac{3ax^2 + 54x - 27a}{(x^2+9)^2}$

PTH $f'(x) = 0 \Leftrightarrow 3ax^2 + 54x - 27a = 0$
$\Leftrightarrow 3(ax^2 + 18x - 9a) = 0$
$\Delta = 18^2 + 4a^2 \cdot 9 = 324 + 36a^2 > 0 \ \forall a \in \mathbb{R}^*$
Alors l'équation $f'(x) = 0$ possède toujours 2 solutions => on a toujours 2 PTH

EXERCICE 1.48 $y = ax^2 + bx + c$ $(5;8)$ $(2,2)$ pente -4
On a les informations:
- passe par le point $(5;8)$: $f(5) = 8$.
- —"— —"— —"— —"— $(2;2)$: $f(2) = 2$
- a tangente avec la pente -4 à $x=2$: $f'(2) = -4$.

$f'(x) = 2ax + b$

$$\begin{cases} f(5) = 8 \Leftrightarrow 25a + 5b + c = 8 & (1) \\ f(2) = 2 \Leftrightarrow 4a + 2b + c = 2 & (2) \\ f'(2) = -4 \Leftrightarrow 4a + b = -1 & (3) \end{cases}$$

(3): $b = -4a - 1$. (4)

(1) $\overset{(4)}{\Leftrightarrow} 25a + 5(-4a-1) + c = 8 \Leftrightarrow$
$\Leftrightarrow 25a - 20a - 5 + c = 8 \Leftrightarrow 5a + c = 13$.

(2) $\overset{(4)}{\Leftrightarrow} 4a + 2(-4a-1) + c = 2 \Leftrightarrow 4a - 8a - 2 + c = 2 \Leftrightarrow$
$\Leftrightarrow -4a + c = 4 \Leftrightarrow c = 4 + 4a$ (5)

$$\begin{array}{rl} & -4a + c = 4 \\ - & 5a + c = 13 \\ \hline & -9a = -9 \Leftrightarrow \underline{a = 1} \end{array}$$

(4) => $b = -4 - 1 \Leftrightarrow \underline{b = -5}$
(5) $\Leftrightarrow c = 4 + 4 = \underline{8 = c}$

EXERCICE 1.49 1) Intersection Ox
$y = x^2 - 1 = 0 \Leftrightarrow x_1 = 1 \quad x_2 = -1$
$y' = f'(x) = 2x$
- $x = 1$. $f'(1) = 2$ $u = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix} // Ox$

On utilise la formule: $\cos\alpha = \dfrac{u \cdot v}{\|u\| \cdot \|v\|}$

$u \cdot v = 1 \cdot 1 + 2 \cdot 0 \Rightarrow u \cdot v = 1$

$\|u\| = \sqrt{1+2^2} = \sqrt{5} \qquad \|v\| = 1$

$\cos\alpha = \dfrac{1}{\sqrt{5}} \Rightarrow \underline{\alpha = 63.43^\circ}$

$\cdot x = -1 \quad f'(-1) = -2 \qquad u = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$u \cdot v = 1 \qquad \|u\| = \sqrt{5} \qquad \|v\| = 1$

$\cos\alpha = \dfrac{1}{\sqrt{5}} \Rightarrow \alpha = 63.43^\circ$

2) $y = x^4 - 5x^2 + 4 \qquad y' = 4x^3 - 10x$

$\cdot \cap Ox$: $y = 0 \Leftrightarrow x^4 - 5x^2 + 4 = 0$ On pose $x^2 = t$

$t^2 - 5t + 4 = 0 \Leftrightarrow t_1 = 4 \quad t_2 = 1$

$x^2 = 4 \Leftrightarrow x = \pm 2 \qquad x^2 = 1 \Leftrightarrow x = \pm 1$

$x = 2$: $f'(2) = 32 - 20 = 12 \qquad u = \begin{pmatrix} 1 \\ 12 \end{pmatrix} \quad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$u \cdot v = 1 \qquad \|u\| = \sqrt{1+12^2} = \sqrt{145} \qquad \|v\| = 1$

$\cos\alpha = \dfrac{1}{\sqrt{145}} \Rightarrow \underline{\alpha = 85.24^\circ}$

$x = -2 \quad f'(2) = -32 + 20 = -12 \qquad u = \begin{pmatrix} 1 \\ -12 \end{pmatrix} \quad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\cos\alpha = \dfrac{1}{\sqrt{1+12^2}} \Rightarrow \underline{\alpha = 85.24^\circ}$

$x = 1 \quad f'(1) = 4 - 10 = -6. \qquad u = \begin{pmatrix} 1 \\ -6 \end{pmatrix} \quad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$u \cdot v = 1 \qquad \|u\| = \sqrt{1+36} = \sqrt{37} \qquad \|v\| = 1$

$\cos\alpha = \dfrac{1}{\sqrt{37}} \Rightarrow \underline{\alpha = 80.54^\circ}$

$x = -1 \quad \underline{\alpha = 80.54^\circ}$

3) $y = f(x) = \dfrac{x}{\sqrt[3]{x^2-1}}$

$$f'(x) = \frac{\sqrt[3]{x^2-1} - x\left(\frac{2x}{3\sqrt[3]{(x^2-1)^2}}\right)}{\sqrt[3]{(x^2-1)^2}} = \frac{3(x^2-1) - 2x^2}{\left(\sqrt[3]{(x^2-1)^2}\right)^2 \cdot 3} = \frac{x^2-3}{3(x^2-1)\sqrt[3]{x^2-1}}$$

On utilise la formule: $\cos\alpha = \dfrac{u \cdot v}{\|u\| \cdot \|v\|}$

$u \cdot v = 1 \cdot 1 + 2 \cdot 0 \Rightarrow u \cdot v = 1$

$\|u\| = \sqrt{1+2^2} = \sqrt{5} \qquad \|v\| = 1$

$\cos\alpha = \dfrac{1}{\sqrt{5}} \Rightarrow \alpha = 63.43°$

$\cdot x = -1 \quad f'(-1) = -2 \qquad u = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$u \cdot v = 1 \qquad \|u\| = \sqrt{5} \qquad \|v\| = 1$

$\cos\alpha = \dfrac{1}{\sqrt{5}} \Rightarrow \alpha = 63.43°$

2) $y = x^4 - 5x^2 + 4 \qquad y' = 4x^3 - 10x$

$\cdot \cap Ox$: $y = 0 \Leftrightarrow x^4 - 5x^2 + 4 = 0$ On pose $x^2 = t$

$t^2 - 5t + 4 = 0 \Leftrightarrow t_1 = 4 \quad t_2 = 1$

$x^2 = 4 \Leftrightarrow x = \pm 2 \qquad x^2 = 1 \Leftrightarrow x = \pm 1$

$\cdot x = 2$: $f'(2) = 32 - 20 = 12 \qquad u = \begin{pmatrix} 1 \\ 12 \end{pmatrix} \quad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$u \cdot v = 1 \qquad \|u\| = \sqrt{1+12^2} = \sqrt{145} \qquad \|v\| = 1$

$\cos\alpha = \dfrac{1}{\sqrt{145}} \Rightarrow \alpha = 85.24°$

$x = -2 \quad f'(2) = -32 + 20 = -12 \qquad u = \begin{pmatrix} 1 \\ -12 \end{pmatrix} \quad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\cos\alpha = \dfrac{1}{\sqrt{1+12^2}} \Rightarrow \alpha = 85.24°$

$x = 1 \quad f'(1) = 4 - 10 = -6 \qquad u = \begin{pmatrix} 1 \\ -6 \end{pmatrix} \quad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$u \cdot v = 1 \qquad \|u\| = \sqrt{1+36} = \sqrt{37} \qquad \|v\| = 1$

$\cos\alpha = \dfrac{1}{\sqrt{37}} \Rightarrow \alpha = 80.54°$

$x = -1 \quad \alpha = 80.54°$

3) $y = f(x) = \dfrac{x}{\sqrt[3]{x^2-1}}$

$$f'(x) = \frac{\sqrt[3]{x^2-1} - x\left(\dfrac{2x}{3\sqrt[3]{(x^2-1)^2}}\right)}{\sqrt[3]{(x^2-1)^2}} = \frac{3(x^2-1) - 2x^2}{\left(\sqrt[3]{(x^2-1)^2}\right)^2 \cdot 3} = \frac{x^2-3}{3(x^2-1)\sqrt[3]{x^2-1}}$$

$\cap Ox$: $y = 0 \Leftrightarrow x = 0$

$f'(0) = -1 \quad u = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \|u\| = \sqrt{2} \quad \|v\| = 1$

$\cos\alpha = \frac{1}{\sqrt{2}} \Rightarrow \underline{\alpha = 45^\circ}$

4) $y = \frac{x^2-4}{x^2+1}$

$y' = \frac{2x(x^2+1)-(x^2-4)\cdot 2x}{(x^2+1)^2} = \frac{\cancel{2x^3}+2x-\cancel{2x^3}+8x}{(x^2+1)^2} = \frac{10x}{(x^2+1)^2}$

$\cap Ox$: $y = 0 \Leftrightarrow x^2-4 = 0 \Leftrightarrow x_1 = 2 \quad x_2 = -2$

$f'(2) = \frac{20}{25} = \frac{4}{5} \quad u = \begin{pmatrix} 1 \\ 4/5 \end{pmatrix} \quad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\|u\| = \sqrt{1+\frac{16}{25}} = \sqrt{\frac{41}{25}} = \frac{\sqrt{41}}{5} \quad \|v\| = 1$

$\cos\alpha = \frac{1}{\sqrt{41}/5} = \frac{5}{\sqrt{41}} \Rightarrow \underline{\alpha = 38.66^\circ}$

<u>EXERCICE 1.50</u> 1) $y = x^2 \quad y = \frac{x^2}{4}+3$

$x^2 = \frac{x^2}{4}+3 \Leftrightarrow 4x^2 = x^2+12 \Leftrightarrow 3x^2 = 12 \Leftrightarrow x^2 = 4 \Leftrightarrow$

$x = \pm 2$

$f'(x) = 2x \quad g'(x) = \frac{1}{2}x$

$\underline{x=2}$ $f'(2) = 4 \quad g'(2) = 1$

$u = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \quad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad u\cdot v = 1+4 = 5$

$\|u\| = \sqrt{17} \quad \|v\| = \sqrt{2}$

$\cos\alpha = \frac{5}{\sqrt{17}\cdot\sqrt{2}} \Rightarrow \underline{\alpha = 30.96^\circ}$

$\underline{x=-2}$ $f'(-2) = -4 \quad g'(-2) = -1$

$u = \begin{pmatrix} 1 \\ -4 \end{pmatrix} \quad v = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad u\cdot v = 1+4 = 5$

$\|u\| = \sqrt{17} \quad \|v\| = \sqrt{2} \Rightarrow \cos\alpha = \frac{5}{\sqrt{17}\sqrt{2}} \Rightarrow \underline{\alpha = 30.96^\circ}$

2) $f(x) = \sin x \quad g(x) = \cos x$

$\sin x = \cos x \Leftrightarrow \frac{\sin x}{\cos x} = 1 \Rightarrow \tan x = 1$

$\Rightarrow x = 45^\circ$ ou $x = 45^\circ + 180^\circ = 225^\circ$

$f'(x) = \cos x \quad g'(x) = -\sin x$

$\underline{x = \frac{\pi}{4}}$ $f'(\pi/4) = \frac{\sqrt{2}}{2} \quad g'(\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}$

$u = \begin{pmatrix} 1 \\ \sqrt{2}/2 \end{pmatrix} \quad v = \begin{pmatrix} 1 \\ -\sqrt{2}/2 \end{pmatrix} \quad u\cdot v = 1-\frac{2}{4} = 1/2$

$\|u\| = \sqrt{3/2} = \|v\|$

$$\cos\alpha = \frac{|u\cdot v|}{\|u\|\cdot\|v\|} = \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{3} \Rightarrow \underline{\alpha = 70.53^\circ}$$

$\underline{x_0 = \frac{5\pi}{4}}$ $\quad f'\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2} \qquad g'\left(\frac{5\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$$u = \begin{pmatrix} 1 \\ -\sqrt{2}/2 \end{pmatrix} \qquad v = \begin{pmatrix} 1 \\ \sqrt{2}/2 \end{pmatrix} \quad u\cdot v = \frac{1}{2}$$

$$\|u\| = \|v\| = \sqrt{3/2}$$

$$\cos\alpha = \frac{1}{3} \Rightarrow \underline{\alpha = 70.53^\circ}$$

3) $f(x) = x^3 - 4x \qquad g(x) = x^3 - 2x^2$

$f(x) = g(x) \Leftrightarrow x^3 - 4x = x^3 - 2x^2 \Leftrightarrow 2x^2 - 4x = 0 \Leftrightarrow$

$2x(x-2) = 0 \Leftrightarrow x = 0 \quad x = 2$

$f'(x) = 3x^2 - 4 \qquad g'(x) = 3x^2 - 4x$

$\underline{x_0 = 0}$ $\quad f'(0) = -4 \qquad g'(0) = 0$

$$u = \begin{pmatrix} 1 \\ -4 \end{pmatrix} \qquad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad u\cdot v = 1$$

$$\|u\| = \sqrt{17} \qquad \|v\| = 1$$

$$\cos\alpha = \frac{u\cdot v}{\|u\|\cdot\|v\|} = \frac{1}{\sqrt{17}} \Rightarrow \underline{\alpha = 75.96}$$

$\underline{x_0 = 2}$ $\quad f'(2) = 8 \qquad g'(2) = 4$

$$u = \begin{pmatrix} 1 \\ 8 \end{pmatrix} \qquad v = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \quad u\cdot v = 1 + 32 = 33$$

$$\|u\| = \sqrt{1+64} = \sqrt{65} \qquad \|v\| = \sqrt{1+4^2} = \sqrt{17}$$

$$\cos\alpha = \frac{33}{\sqrt{65}\cdot\sqrt{17}} \Rightarrow \underline{\alpha = 6.9^\circ}$$

<u>EXERCICE 1.51</u> 1) $f(x) = x^2 \quad g(x) = \frac{1}{2} - ax^2$

$m_1 = f'(x) = 2x \qquad g'(x) = -2ax = m_2$

$m_1 \cdot m_2 = -1 \Leftrightarrow 2x_0 \cdot (-2ax_0) = -1 \Leftrightarrow -4ax_0^2 = -1$ ①

$f(x_0) = g(x_0) \Leftrightarrow x_0^2 = \frac{1}{2} - ax_0^2 \Leftrightarrow (a+1)x_0^2 = \frac{1}{2}$ ②

$$\begin{cases} 4ax_0^2 = 1 \\ (a+1)x_0^2 = \frac{1}{2} \end{cases} \xRightarrow{x_0 \neq 0} \frac{4a}{a+1} = \frac{1}{\frac{1}{2}} = 2 \Leftrightarrow$$

$\Leftrightarrow 4a = 2a + 2 \Leftrightarrow 2a = 2 \Leftrightarrow \underline{a = 1}$

① $\Leftrightarrow 4x_0^2 = 1 \Leftrightarrow x_0 = \pm\frac{1}{2} \qquad a = 1$

2) $f(x) = ax^2$ $g(x) = \dfrac{1-x^2}{a}$

$f'(x) = 2ax$ $g'(x) = -\dfrac{2x}{a}$

$f(x) = g(x) \Leftrightarrow ax^2 = \dfrac{1-x^2}{a} \Leftrightarrow a^2x^2 + x^2 = 1 \Leftrightarrow$

$\Leftrightarrow (a^2+1)x^2 = 1$ ①

$m_1 \cdot m_2 = -1 \Leftrightarrow 2\cancel{a}x \cdot \left(-\dfrac{2x}{\cancel{a}}\right) = -1 \Leftrightarrow$

$\Leftrightarrow -4x^2 = -1 \Leftrightarrow x^2 = \dfrac{1}{4} \Leftrightarrow x = \pm\dfrac{1}{2}$

- $x = \dfrac{1}{2}$ ① $\Rightarrow (a^2+1)\dfrac{1}{16} = 1 \Leftrightarrow a^2+1 = 16 \Leftrightarrow$
  $a^2 = 15 \Leftrightarrow \underline{a = \pm\sqrt{15}}$
- $x = -\dfrac{1}{2}$ ① $\Rightarrow (a^2+1)\dfrac{1}{16} = 1 \Leftrightarrow \underline{a = \pm\sqrt{15}}$

3) $f(x) = 2x^2 - a$ $g(x) = \dfrac{x^2}{a}$

$f'(x) = 4x$ $g'(x) = \dfrac{2x}{a}$

$f(x) = g(x) \Rightarrow 2x^2 - a = \dfrac{x^2}{a} \Leftrightarrow 2ax^2 - a^2 = x^2 \Leftrightarrow$

$\Leftrightarrow (2a-1)x^2 = a^2$ ①

$m_1 \cdot m_2 = -1 \Leftrightarrow 4x \cdot \dfrac{2x}{a} = -1 \Leftrightarrow$

$\Leftrightarrow 8x^2 = -a \Leftrightarrow x^2 = -\dfrac{a}{8}$ (2)

① donne $(2a-1)\left(-\dfrac{a}{8}\right) = a^2 \Leftrightarrow$

$\Leftrightarrow (2a-1)(-a) = 8a^2 \Leftrightarrow$

$\Leftrightarrow -2a^2 + a = 8a^2 \Leftrightarrow 10a^2 - a = 0$

$a(10a-1) = 0 \Leftrightarrow a = 0 \quad a = 1/10$

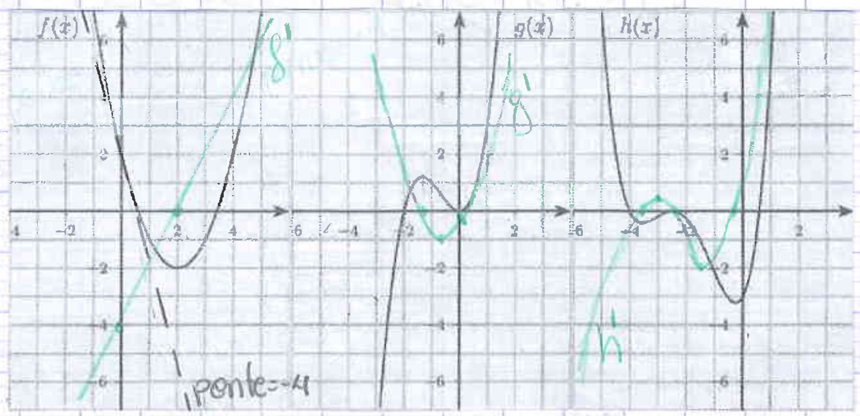
- Si $a = 0$ g n'est pas definie (imp)
- Si $a = \dfrac{1}{10}$ g (2) donne $x^2 = -\dfrac{1}{80}$ (imp)

EXERCICE 1.52 En $x \approx -0.8$ on a un point à tangente horizontale (sommet de la parabole)
La dérivée est donc 0 en ce point-là
Pour $x < -0.8$ la fontion est decroissante
$\Rightarrow f' < 0$

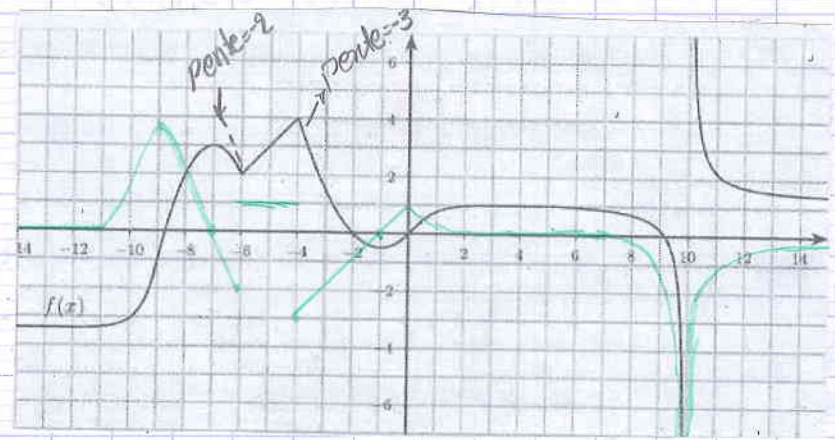
Pour $x > -0.8$ la fonction est croissante $\Rightarrow f' > 0$

La seule illustration qui correspond à cela est la première

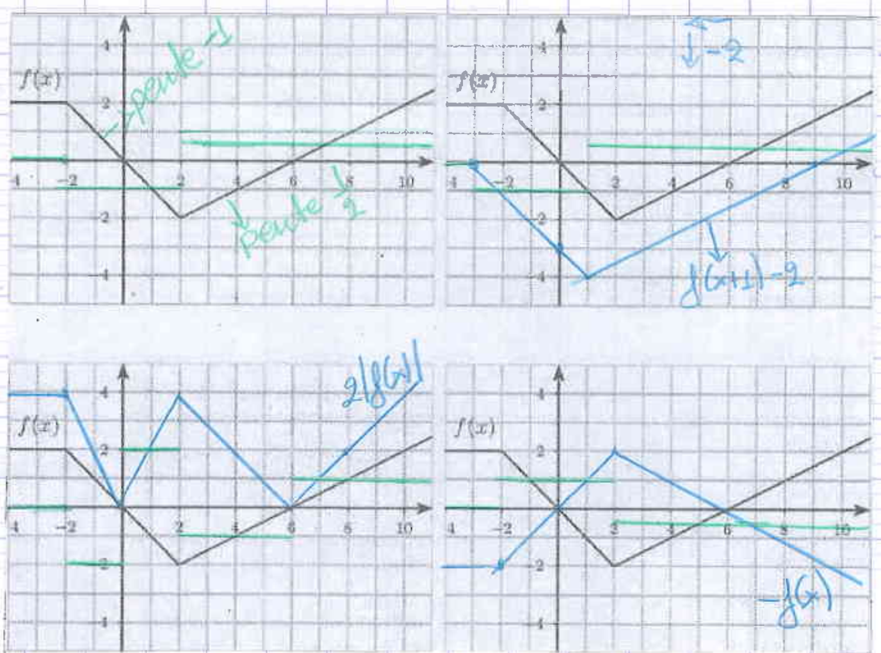
Exercice 1.53



Exercice 1.54



Exercice 1.55



EXERCICE 1.56

$x, y > 0 \quad x + y = 10 \Rightarrow y = 10 - x$

$f(x) = x^3 \cdot (10 - x)^2$

$f'(x) = 3x^2(10-x)^2 - x^3 \cdot 2(10-x) =$

$= (10-x) \cdot x^2 (3(10-x) - 2x) =$

$= (10-x) x^2 (30 - 3x - 2x) =$

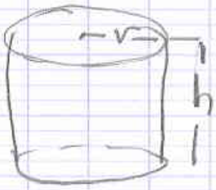
$= (10-x) x^2 (30 - 5x) = 5x^2(10-x)(6-x).$

$f'(x) = 0 \Leftrightarrow x = 10 \quad x = 0 \quad x = 6.$

x	0		6		10
f'		+	0	−	
f		↗	Max	↘	

$x = 6 \quad f(6) = 3456$

EXERCICE 1.57



$V = \pi r^2 \cdot h = 500 \Rightarrow h = \frac{500}{\pi r^2}$

$A_t = 2 \cdot \pi r^2 + 2\pi \cdot r \cdot h$

$A(r) = 2\pi r^2 + 2\cancel{\pi} \cancel{r} \cdot \frac{500}{\cancel{\pi} r^{\cancel{2}}} \Rightarrow A(r) = 2\pi r^2 + \frac{1000}{r}$

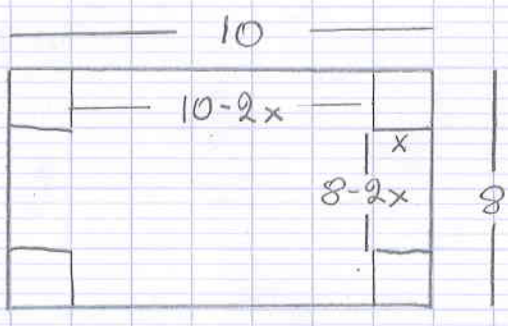
$A'(r) = 4\pi r - \frac{1000}{r^2} = 0 \Leftrightarrow 4\pi r = \frac{1000}{r^2} \Leftrightarrow$

$\Leftrightarrow 4\pi r^3 = 1000 \Leftrightarrow r = \sqrt[3]{\frac{1000}{4\pi}} \approx 4.3$

r	0		4.3		$+\infty$
A'		−	0	+	
A		↘	Min	↗	

$h = \frac{500}{\pi \, 4.3^2} = 8.61$

EXERCICE 1.58



$V = (10-2x)(8-2x) \cdot x = (80 - 16x - 20x + 4x^2)x$

$= (80 - 36x + 4x^2)x = 4x^3 - 36x^2 + 80x$

$V'(x) = 12x^2 - 72x + 80$

$V'(x) = 0 \Leftrightarrow 4(3x^2 - 18x + 20) = 0$

$\Delta = 84 \quad x_{1,2} = \dfrac{18 \pm \sqrt{84}}{6} = \begin{cases} x_1 = 1.47 \\ x_2 = 4.53 \end{cases}$

x	0		$1{,}47$		$4{,}53$		$+\infty$
V'		$+$	0	$-$	0	$+$	
V		$\nearrow$	Max	$\searrow$		$\nearrow$	

$x = 1{,}47$

$V \approx 52{,}51$

Exercice 1.59

1) $2y + x = 200 \Leftrightarrow x = 200 - 2y$

$A = x \cdot y \Rightarrow A(y) = (200 - 2y)y = -2y^2 + 200y$

$A'(y) = -4y + 200 = 0 \Leftrightarrow y = 50$

y	0		50	
A'		$+$	0	$-$
A		$\nearrow$	Max	$\searrow$

$y = 50 \qquad x = 200 - 100 \Rightarrow x = 100$

2) $A = x \cdot y = 300 \Rightarrow y = \dfrac{300}{x}$

$L = 2y + x \Rightarrow L(x) = 2 \cdot \dfrac{300}{x} + x \Rightarrow$

$\Rightarrow L(x) = \dfrac{600}{x} + x \Rightarrow L'(x) = -\dfrac{600}{x^2} + 1$

$L'(x) = 0 \Leftrightarrow -\dfrac{600}{x^2} + 1 = 0 \Leftrightarrow \dfrac{600}{x^2} = 1 \Leftrightarrow x^2 = 600$

$\Rightarrow x = \pm\sqrt{600} = \pm 10\sqrt{6}$

x	0		$10\sqrt{6}$		$+\infty$
L'		$-$	0	$+$	
L		$\searrow$	Min	$\nearrow$	

Min: $x = 10\sqrt{6}$

$y = \dfrac{300}{10\sqrt{6}} = \dfrac{30}{\sqrt{6}}$